CONTINUOUS MAPPINGS ON SUBSPACES OF PRODUCTS WITH THE κ-BOX TOPOLOGY

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Abstract. Much of General Topology addresses this issue: Given a function $f \in C(Y, Z)$ with $Y \subseteq Y'$ and $Z \subseteq Z'$, find $\overline{f} \in C(Y', Z')$, or at least $\overline{f} \in C(\mathrm{cl}_{Y'} Y, Z')$, such that $f \subseteq \overline{f}$; sometimes $Z = Z'$ is demanded. In this spirit the authors prove several quite general theorems in the context $Y' = (X_I)_\kappa = \Pi_{i\in I} X_i$ in the κ-box topology (that is, with basic open sets of the form $\Pi_{i\in I} U_i$ with U_i open in X_i and with $U_i \neq X_i$ for $< \kappa$-many $i \in I$). A representative sample result, extending to the κ-box topology some results of Comfort and Negrepontis, of Noble and Ulmer, and of Hušek, is this.

Theorem. Let $\omega \leq \kappa << \alpha$ (that means: $\kappa < \alpha$, and $[\beta < \alpha$ and $\lambda < \kappa] \Rightarrow \beta^\lambda < \alpha$) with α regular, $\{X_i : i \in I\}$ be a set of non-empty spaces with each $d(X_i) < \alpha$, $\pi[Y] = X_J$ for each non-empty $J \subseteq I$ such that $|J| < \alpha$, and the diagonal in Z be the intersection of $< \alpha$-many regular-closed subsets of $Z \times Z$. Then (a) Y is pseudo-(α, α)-compact, (b) for every $f \in C(Y, Z)$ there is $J \in [I]^{<\alpha}$ such that $f(x) = f(y)$ whenever $x_J = y_J$, and (c) every such f extends to $\overline{f} \in C((X_I)_\kappa, Z)$.

2000 *Mathematics Subject Classification.* Primary 54B10, 54D35; Secondary 54A35.

Key words and phrases. κ-box topology; pseudo-(α, κ)-compact; κ-invariant set; Σ_κ-product; calibre; Souslin number; functional dependence; continuous extension of function.

Honoring Neil Hindman and his many contributions to our profession.

The second author expresses his gratitude to Wesleyan University for hospitality and support during his sabbatical in the spring semester, 2008.

1. Introduction

Many papers in set-theoretic topology engage specific instances of this general question: Given spaces X, Y and Z with $Y \subseteq X$, and given continuous $f : Y \rightarrow Z$, does f extend to continuous $\overline{f} : X \rightarrow Z$? The case where X is a product, in the usual product topology or (more recently) in a κ-box topology, has received particular attention; typically the issue then focuses on whether f "factors through" or "depends upon" some manageable (small) set of coordinates. This paper is in that tradition. We use tools, now fully absorbed into the culture and viewed as standard or routine, introduced initially and exploited by such workers as Mazur, Gleason, Miščenko, and Engelking, for the usual product topology, and by Comfort and Negrepontis for the more general case of κ-box topology. Later Hušek, for the usual product topology, developed sharper ideas and isolated near-minimal conditions on X, Y, Z and f which guarantee such an extension. In this paper we extend some of the ideas and results of Hušek and Comfort and Negrepontis to the κ-box topology. As this paper progresses, we specify the contributions of these and other workers *in situ*. In some cases, our arguments are again routine adaptations of now-familiar arguments to the new contexts we consider, so some portions of this paper (by no means all) may be viewed as confirmation of unsurprising generalizations of classical results.

Notation and Definitions 1.1. (a) Topological spaces considered here are not subjected to any special standing separation properties. Additional hypotheses are imposed as required.

(b) ω is the least infinite cardinal, and α, β, γ and κ are infinite cardinals.

The notation $\kappa << \alpha$ means that α *is strongly* κ*-inaccessible*. That is: $\kappa < \alpha$, and $\beta^\lambda < \alpha$ whenever $\beta < \alpha$ and $\lambda < \kappa$ (see [5, p. 254]).

For I a set, write $[I]^\alpha := \{J \subseteq I : |J| = \alpha\}$; the notations $[I]^{<\alpha}$, $[I]^{\leq\alpha}$ are defined analogously.

(c) A (not necessarily faithfully) indexed family $\{A_i : i \in I\}$ of non-empty subsets of a space X is *locally* $< \kappa$ if there is an open cover $\mathcal{U}$ of X such that $|\{i \in I : U \cap A_i \neq \emptyset\}| < \kappa$ for each $U \in \mathcal{U}$. A space $X = (X, \mathcal{T})$ is *pseudo*-(α, κ)-*compact* if every indexed locally $< \kappa$ family $\{U_i : i \in I\} \subseteq \mathcal{T} \backslash \{\emptyset\}$ satisfies $|I| < \alpha$. In this terminology, the familiar pseudocompact spaces are the pseudo-(ω, ω)-compact spaces.

(d) For a set $\{X_i : i \in I\}$ of sets and $J \subseteq I$, we write $X_J := \Pi_{i \in J} X_i$; and for a *generalized rectangle* $A = \Pi_{i \in I} A_i \subseteq X_I$ the *restriction set of* A, denoted $R(A)$, is the set $R(A) = \{i \in I : A_i \neq X_i\}$. When each $X_i = (X_i, \mathcal{T}_i)$ is a space, the symbol $(X_I)_\kappa$ denotes X_I with the κ-*box topology*; this is the topology for which $\{\Pi_{i \in I} U_i : U_i \in \mathcal{T}_i, |R(U)| < \kappa\}$ is a base. Thus the ω-box topology on X_I is the usual product topology. We note that even when κ is regular, the intersection of fewer than κ-many sets, each open in $(X_I)_\kappa$, may fail to be open in $(X_I)_\kappa$.

(e) For $x, y \in Y \subseteq X_I$ the *difference set* $d(x, y)$ is the set $d(x, y) := \{i \in I : x_i \neq y_i\}$.

(f) For $p \in X_I = \Pi_{i \in I} X_i$, the κ-Σ-*product of* X_I *based at* p is the set $\Sigma_\kappa(p) := \{x \in X_I : |d(p, x)| < \kappa\}$.

(g) A set $Y \subseteq X_I$ is κ-*invariant* [16] provided that for every $x, y \in Y$ and $J \in [I]^{<\kappa}$ the point $z \in X_I$ defined by $z_J = x_J$, $z_{I \backslash J} = y_{I \backslash J}$, is in Y.

(h) For spaces Y and Z we denote by $C(Y, Z)$ the set of continuous functions from Y into Z.

Remarks 1.2. (a) In the notation of 1.1(f), the usual Σ-product based at $p \in X_I$ is the set $\Sigma(p) = \Sigma_{\omega^+}(p)$, and the "little σ-product" [6] is the set $\sigma(p) := \Sigma_\omega(p) \subseteq X_I$. If $|X_i| \geq 2$ for all $i \in I$ then $\pi_J[\Sigma_\kappa(p)] = X_J$ if and only if $J \in [I]^{<\kappa}$, so if the sets X_i are topological spaces then each $\Sigma_\kappa(p) \subseteq X_I$ is dense in $(X_I)_\kappa$.

(b) Clearly, every Σ_κ-space in X_I, also every generalized rectangle in X_I, is κ-invariant. (Note that the notion κ-invariant is closely related to, but different from, the notion of a subspace *invariant under projection* defined in [20].)

2. Generalizing Lemma 10.1 from [5]

An error or gap in the proof of Lemma 10.1 in [5], perhaps deriving from an apparently harmless ambiguity in [3], left undetermined the truth-status of that lemma, of the several results in Chapter 10 of [5] which depended upon it, and of several theorems published subsequently by authors who trustingly cited [3] and [5, 10.1]. A full, correct, and detailed proof of (a generalization of) that lemma finally appeared in [2]. Theorem 2.2 below, which powers the applications in our subsequent sections, generalizes further [5, Lemma 10.1] and [3]. Among the present improvements is the fact that our spaces X_i $(i \in I)$ and Z are not subjected to any separation axiom. This contrasts with the treatments in [2], where Z is assumed metrizable, and in [5], where all spaces are also assumed to be Tychonoff.

Notation 2.1. For a space Z let $\Delta_Z := \{(z,z) \in Z \times Z : z \in Z\}$ be the *diagonal* of Z.

Theorem 2.2. *Let $\omega \leq \kappa \leq \alpha$ with either $\kappa < \alpha$ or α regular, $\{X_i : i \in I\}$ be a set of non-empty spaces, Y be a pseudo-(α,κ)-compact subspace of $(X_I)_\kappa$ which is dense in some open subset of $(X_I)_\kappa$, and Z be a space. Then for every open neighborhood O of Δ_Z in $Z\times Z$ and for every $f \in C(Y,Z)$, there is $J \in [I]^{<\alpha}$ such that $(f(x), f(y)) \in \overline{O}$ whenever $x, y \in Y$ are such that $x_J = y_J$.*

Proof. We suppose the result fails.

Let $Y \subset U \subset \overline{Y}$, U open in $(X_I)_\kappa$. For each $\xi < \alpha$ we define $x(\xi), y(\xi) \in Y$, basic open neighborhoods $U(\xi) \subset U$ and $V(\xi) \subset U$ in $(X_I)_\kappa$ of $x(\xi)$ and $y(\xi)$, respectively, and $J(\xi), A(\xi) \subseteq I$ such that:

(i) $(f(x), f(y)) \notin \overline{O}$ if $x \in U(\xi) \cap Y$, $y \in V(\xi) \cap Y$;
(ii) $A(\xi) := \{i \in R(U(\xi)) \cup R(V(\xi)) : x(\xi)_i \neq y(\xi)_i\}$;
(iii) $U(\xi)_i = V(\xi)_i$ if $i \in I\backslash A(\xi)$;
(iv) $x(\xi)_i = y(\xi)_i$ for $i \in J(\xi)$; and further with
(v) $J(0) = \emptyset$, $J(\xi) = \cup_{\eta<\xi} A(\eta)$ for $0 < \xi < \alpha$.

To begin, we choose $x(0) \in Y$ and $y(0) \in Y$ such that $(f(x(0)), f(y(0))) \notin \overline{O}$, and open neighborhoods $W_x(0)$ and $W_y(0)$ in Z of $f(x(0))$ and $f(y(0))$, respectively, such that $(W_x(0) \times W_y(0)) \cap \overline{O} = \emptyset$. Then $(W_x(0) \times W_y(0)) \cap \Delta_Z = \emptyset$, so $W_x(0) \cap W_y(0) = \emptyset$. It follows from the continuity of f that there are disjoint, basic open neighborhoods $\widetilde{U(0)} \subset U$ and $\widetilde{V(0)} \subset U$ in $(X_I)_\kappa$ of $x(0)$ and $y(0)$, respectively, such that $(f(x), f(y)) \notin \overline{O}$ for all $x \in \widetilde{U(0)} \cap Y$ and $y \in \widetilde{V(0)} \cap Y$. Then, define $A(0) := \{i \in R(\widetilde{U(0)}) \cup R(\widetilde{V(0)}) : x(0)_i \neq y(0)_i\}$ and define (basic open) neighborhoods $U(0)$ and $V(0)$ in $(X_I)_\kappa$ of $x(0)$ and $y(0)$, respectively, as follows:

$U(0)_i = V(0)_i = X_i$ if $i \in I \backslash (R(\widetilde{U(0)}) \cup R(\widetilde{V(0)}))$;
$U(0)_i = V(0)_i = \widetilde{U(0)}_i \cap \widetilde{V(0)}_i$ if $i \in (R(\widetilde{U(0)}) \cup R(\widetilde{V(0)})) \backslash A(0)$; and
$U(0)_i = \widetilde{U(0)}_i$, $V(0)_i = \widetilde{V(0)}_i$ if $i \in A(0)$.

Then $U(0) \subseteq U$ and $V(0) \subseteq U$, and (i)–(v) hold for $\xi = 0$.

Suppose now that $0 < \xi < \alpha$ and that $x(\eta), y(\eta) \in Y$, $U(\eta) \subset U$, $V(\eta) \subset U$, and $A(\eta), J(\eta) \subseteq I$ have been defined for $\eta < \xi$ satisfying (the analogues of) (i)–(v). Since $J(\xi)$, defined by (v), satisfies $|J(\xi)| < \alpha$, there are $x(\xi)$ and $y(\xi)$ in Y such that (iv) holds and $(f(x(\xi)), f(y(\xi))) \notin \overline{O}$, and open neighborhoods $W_x(\xi)$ and $W_y(\xi)$ in Z of $f(x(\xi))$ and $f(y(\xi))$, respectively, such that $(W_x(\xi) \times W_y(\xi)) \cap \overline{O} = \emptyset$. Then $(W_x(\xi) \times W_y(\xi)) \cap \Delta_Z = \emptyset$, so $W_x(\xi) \cap W_y(\xi) = \emptyset$. It follows from the continuity of f that there are disjoint, basic open neighborhoods $\widetilde{U(\xi)} \subset U$ and $\widetilde{V(\xi)} \subset U$ in $(X_I)_\kappa$ of $x(\xi)$ and $y(\xi)$, respectively, such that $(f(x), f(y)) \notin \overline{O}$ for all $x \in \widetilde{U(\xi)} \cap Y$, $y \in \widetilde{V(\xi)} \cap Y$. Then, define $A(\xi) := \{i \in R(\widetilde{U(\xi)}) \cup R(\widetilde{V(\xi)}) : x(\xi)_i \neq y(\xi)_i\}$ and define (basic open) neighborhoods $U(\xi)$ and $V(\xi)$ in $(X_I)_\kappa$ of $x(\xi)$ and $y(\xi)$, respectively, as follows:

$U(\xi)_i = V(\xi)_i = X_i$ if $i \in I \backslash (R(\widetilde{U(\xi)}) \cup R(\widetilde{V(\xi)}))$;

$U(\xi)_i = V(\xi)_i = \widetilde{U(\xi)}_i \cap \widetilde{V(\xi)}_i$ if $i \in (R(\widetilde{U(\xi)}) \cup R(\widetilde{V(\xi)}))\backslash A(\xi)$; and

$U(\xi)_i = \widetilde{U(\xi)}_i$, $V(\xi)_i = \widetilde{V(\xi)}_i$ if $i \in A(\xi)$.

Then $U(\xi) \subseteq U$ and $V(\xi) \subseteq U$, and (i)–(v) hold. The recursive definitions are complete.

We note that if $\eta < \xi < \alpha$ and $i \in A(\eta)$ then $x(\xi)_i = y(\xi)_i$ and hence $i \notin A(\xi)$. That is: the sets $A(\xi)$ $(\xi < \alpha)$ are pairwise disjoint.

Since the space Y is pseudo-(α, κ)-compact, there is $\overline{p} \in Y$ such that each basic neighborhood $W \subset U$ of $\overline{p}$ in $(X_I)_\kappa$ satisfies $|\{\xi < \alpha : W \cap U(\xi) \neq \emptyset\}| \geq \kappa$. Fix such W and choose $\overline{\xi} < \alpha$ such that $W \cap U(\overline{\xi}) \neq \emptyset$ and no $i \in R(W)$ is in $A(\overline{\xi})$. (This is possible since $|R(W)| < \kappa$ and each $i \in R(W)$ is in at most one of the sets $A(\xi)$.) For each such $\overline{\xi}$ by (iii) we have $U(\overline{\xi})_i = V(\overline{\xi})_i$ for all $i \in R(W)$, so also $W \cap V(\overline{\xi}) \neq \emptyset$.

Since Y is dense in $U \subseteq (X_I)_\kappa$, the previous paragraph shows this: For each neighborhood W in $U \subseteq (X_I)_\kappa$ of $\overline{p}$ there is $\overline{\xi}$ such that $W \cap U(\overline{\xi}) \cap Y \neq \emptyset$ and $W \cap V(\overline{\xi}) \cap Y \neq \emptyset$. Let G be an open neighborhood in Z of $f(\overline{p})$ such that $G \times G \subset O$. Since f is continuous at $\overline{p}$ there is a basic open neighborhood $W' \subset U$ of $\overline{p}$ such that $f[W' \cap Y] \subset G$. Then there is $\overline{\xi}'$ such that $W' \cap U(\overline{\xi}') \cap Y \neq \emptyset$ and $W' \cap V(\overline{\xi}') \cap Y \neq \emptyset$, and with $x \in W' \cap U(\overline{\xi}') \cap Y$ and $y \in W' \cap V(\overline{\xi}') \cap Y$ we have $(f(x), f(y)) \in G \times G \subseteq O$. This contradicts (i), completing the proof. □

Historical Remarks 2.3. Our Theorem 2.2 builds on a long history. Conditions sufficient to ensure that a continuous function $f : X_I = (X_I)_\omega \to Z$ depends on countably many coordinates were established by Mazur [20], Corson and Isbell [7], A. Gleason (see Isbell [17]), and Miščenko [21]. Engelking [8] provided a detailed historical analysis, and improved most of the then-known results with the theorem that every function $f \in C(X_I, Z)$ depends on countably many coordinates if Z is a Hausdorff space for which the diagonal is a G_δ-set in

$Z \times Z$, provided that every subproduct X_F with $|F| < \omega$ is a Lindelöf space. Note that the above-cited papers concern exclusively the dependence of $f \in C(X_I, Z)$ on a small (countable) set of coordinates; no question of extendability arises, since already $\mathrm{dom}(f) = X_I$. To the authors' knowledge, it was Mazur [20] who first considered index-dependence for functions $f : Y \to Z$ defined on subspaces Y of X_I which are what he called *invariant under projection*, a condition satisfied by every subspace of the form $\Sigma_\kappa(p)$. Glicksberg [11] used a recursive argument, a forerunner of ours in Theorem 2.2, to show that if $X = X_I = (X_I)_\omega$ is compact and Y is dense and pseudocompact in X (e.g., if Y contains a set $\Sigma_{\omega^+}(p) \subseteq X_I$) then every $f \in C(Y, \mathbb{R})$ extends continuously over X_I (see also Kister [19] and Noble and Ulmer [22] for similar results). It was a short step [3] to replace the condition $Y \supseteq \Sigma_{\omega^+}(p)$ by the weaker condition that Y projects onto each subproduct X_J with $J \in [I]^{\leq\omega}$, and then to study, for $\alpha \geq \omega$ and $\kappa \geq \omega$, spaces $Y \subseteq (X_I)_\kappa$ which project onto each X_J with $|J| \leq \alpha$. Hušek [13] identified that latter property, which he denoted $V(\alpha)$, as useful and worthy of study in its own right; he found several new results when $\alpha = \omega$ or $\alpha = \omega^+$ and $\kappa = \omega$ [13]. Then in [14], [15], [16], Hušek introduced and studied the concept of a space having (weakly) γ-inaccessible diagonal (see 4.3) and in [16] the concept of a κ-invariant subset of a product space (see 1.1(g)). These fruitful conditions have proved adequate for many of the applications (see Theorems 4.5, 5.3, 5.5 below).

3. Conditions Sufficient that $(X_I)_\kappa$ is Pseudo-(α, β)-compact

In Theorem 2.2 and frequently hereafter we discuss spaces of the form $(X_I)_\kappa$ which for certain cardinals α, β are hypothesized to be pseudo-(α, β)-compact; in many cases that condition is required even of certain specialized dense subspaces Y of $(X_I)_\kappa$. We show in this section (see Theorem 3.4) that these conditions are neither unusual nor artificial: if $\omega \leq \kappa << \alpha$ with α regular and $d(X_i) < \alpha$ for each $i \in I$, then $(X_I)_\kappa$ is even pseudo-(α, α)-compact; further, that condition is inherited by each subspace $Y \subseteq (X)_\kappa$ such that $\pi_J[Y] = X_J$ for every non-empty $J \in [I]^{<\kappa}$ (see Theorem 3.4). The pattern of our arguments follows similar results in [5], especially Theorem 3.6.

As usual, a *cellular family* in a space X is a family of pairwise disjoint, non-empty open subsets of X.

Definition 3.1. Let X be a space.

(a) The *density character* of X is the number $d(X) := \min\{|D| : D \text{ is dense in } X\}$.

(b) The *Souslin number* of X is the number $S(X) := \min\{\alpha :$ every cellular family $\mathcal{U}$ in X satisfies $|\mathcal{U}| < \alpha\}$;

(c) X has *calibre* α if for every (indexed) family $\{U(\xi) : \xi < \alpha\}$ of non-empty open subsets of X there is $A \in [\alpha]^\alpha$ such that $\bigcap_{\xi \in A} U(\xi) \neq \emptyset$.

Remarks 3.2. (a) It is well to observe that a space may have calibre α for various distinct cardinals α. For example, a separable space clearly has calibre α for every regular uncountable cardinal α.

(b) The concept of calibre was introduced, and related to other cardinal invariants with an emphasis on product spaces, in a remarkable sequence of papers by Shanin [23], [24], [25]. Many of Shanin's results are recorded, and in certain cases amplified and generalized, in [5].

We will often use the following facts.

Theorem 3.3. *Let X be a space and $\alpha \geq \omega$.*

(a) *If $d(X) < \mathrm{cf}(\alpha)$ then X has calibre $\mathrm{cf}(\alpha)$ and calibre α;*

(b) *if X has calibre $\mathrm{cf}(\alpha)$ or calibre α then $S(X) \leq \mathrm{cf}(\alpha)$; and*

(c) *if $S(X) \leq \mathrm{cf}(\alpha)$ then every dense subspace of X is pseudo-$(\mathrm{cf}(\alpha), \mathrm{cf}(\alpha))$-compact and pseudo-$(\alpha, \alpha)$-compact.*

Proof. (a) To see that X has calibre α, let $D \in [X]^{<\mathrm{cf}(\alpha)}$ be dense in X and let $\{U(\xi) : \xi < \alpha\}$ be an indexed family of non-empty open subsets of X. For $p \in D$ set $A(p) := \{\xi <$

$\alpha : p \in U(\xi)\}$. Since $\alpha = \bigcup_{p\in D} A(p)$ and $|D| < \mathrm{cf}(\alpha)$, there is $\overline{p} \in D$ such that $|A(\overline{p})| = \alpha$, as required.

That X also has calibre $\mathrm{cf}(\alpha)$ then follows by replacing α by $\mathrm{cf}(\alpha)$ (using $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha)$). (In fact it is noted in [5, 2.3(a)] that every space with calibre α has calibre $\mathrm{cf}(\alpha)$.)

(b) That $S(X) \leq \mathrm{cf}(\alpha)$ if X has calibre $\mathrm{cf}(\alpha)$ is obvious. We show that if $S(X) > \mathrm{cf}(\alpha)$ then X does not have calibre α. Indeed suppose that $\mathrm{cf}(\alpha) < \alpha$ and that $\{V(\eta) : \eta < \mathrm{cf}(\alpha)\}$ is a cellular family in X. Let $\{\alpha_\eta : \eta < \mathrm{cf}(\alpha)\}$ be a set of cardinals such that

$$0 < \alpha_0 < \cdots < \alpha_\eta < \alpha_{\eta+1} < \cdots < \alpha$$

and $\alpha = \sup_{\eta<\mathrm{cf}(\alpha)} \alpha_\eta$. For $\xi < \alpha$ define $U(\xi) = V(\eta)$ if $\alpha_\eta \leq \xi < \alpha_{\eta+1}$. If $A \in [\alpha]^\alpha$ there are (distinct) $\xi', \xi'' < \alpha$ and distinct $\eta', \eta'' < \mathrm{cf}(\alpha)$ such that $\alpha_{\eta'} \leq \xi' < \alpha_{\eta'+1}$ and $\alpha_{\eta''} \leq \xi'' < \alpha_{\eta''+1}$, and then

$$\bigcap_{\xi\in A} U(\xi) \subseteq U(\xi') \cap U(\xi'') = V(\eta') \cap V(\eta'') = \emptyset.$$

Therefore X does not have calibre α.

(c) Evidently the Souslin number is invariant upon passage back and forth between a space and any of its dense subspaces, so it suffices to show that X has the indicated properties.

We show first that X is pseudo-(α, α)-compact.

Suppose that there is a family $\{U(\xi) : \xi < \alpha\}$ of open sets in X which is locally $< \alpha$. Let $\xi_0 = 0$ and $x(0) \in U(\xi_0)$, and choose an open neighborhood $V(0) \subseteq U(\xi_0)$ of $x(0)$ such that $A(0) := \{\xi < \alpha : V(0) \cap U(\xi) \neq \emptyset\}$ satisfies $|A(0)| < \alpha$. Let $\overline{\eta} < \mathrm{cf}(\alpha)$ and suppose for every $\eta < \overline{\eta}$ that $\xi_\eta < \alpha$, $x(\eta) \in U(\xi_\eta)$, and an open neighborhood $V(\eta) \subseteq U(\xi_\eta)$ of $x(\eta)$ such that the set $A(\eta) := \{\xi < \alpha : V(\eta) \cap U(\xi) \neq \emptyset\}$ satisfies $|A(\eta)| < \alpha$ have been selected. Since $\overline{\eta} < \mathrm{cf}(\alpha)$ we have $|\bigcup_{\eta<\overline{\eta}} A(\eta)| < \alpha$ so there is $\xi_{\overline{\eta}} < \alpha$ such that $U(\xi_{\overline{\eta}}) \cap (\bigcup_{\eta<\overline{\eta}} V(\eta)) = \emptyset$. Then choose $x(\overline{\eta}) \in U(\xi_{\overline{\eta}})$ and an open neighborhood $V(\overline{\eta}) \subseteq U(\xi_{\overline{\eta}})$ of $x(\overline{\eta})$ such that $A(\overline{\eta}) := \{\xi < \alpha : V(\overline{\eta}) \cap U(\xi) \neq \emptyset\}$ satisfies $|A(\overline{\eta})| < \alpha$. The recursion is complete. The family $\{V(\eta) : \eta <$

$\mathrm{cf}(\alpha)\}$ is cellular, contrary to the hypothesis $S(X) \leq \mathrm{cf}(\alpha)$. Thus X is pseudo-(α, α)-compact.

That X also is pseudo-$(\mathrm{cf}(\alpha), \mathrm{cf}(\alpha))$-compact then follows by replacing α by $\mathrm{cf}(\alpha)$ (using $\mathrm{cf}(\mathrm{cf}(\alpha) = \mathrm{cf}(\alpha))$. (In fact it is noted in [5, 2.3(c)] that every pseudo-(α, α)-compact space is pseudo-$(\mathrm{cf}(\alpha), \mathrm{cf}(\alpha))$-compact.) □

Theorem 3.4. *Let* $\omega \leq \kappa << \mathrm{cf}(\alpha)$ *and* $\{X_i : i \in I\}$ *be a set of non-empty spaces. Consider the following properties:*

(i) $d(X_i) < \mathrm{cf}(\alpha)$ *for each* $i \in I$*;*

(ii) $d((X_J)_\kappa) < \mathrm{cf}(\alpha)$ *for each non-empty* $J \in [I]^{<\kappa}$*;*

(iii) *each* $(X_J)_\kappa$ *with* $\emptyset \neq J \in [I]^{<\kappa}$ *has calibre* $\mathrm{cf}(\alpha)$ *and calibre* α*;*

(iv) $S((X_J)_\kappa) \leq \mathrm{cf}(\alpha)$ *for each non-empty* $J \in [I]^{<\kappa}$*;*

(v) $S((X_I)_\kappa) \leq \mathrm{cf}(\alpha)$*;*

(vi) *if* $\emptyset \neq J \subset I$ *and* W *is dense in some non-empty open subset of* $(X_J)_\kappa$*, then* W *is pseudo-(*$\mathrm{cf}(\alpha), \mathrm{cf}(\alpha)$*)-compact and pseudo-(*$\alpha, \alpha$*)-compact.*

Then (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (iv), (iv) $\Rightarrow$ (v), *and* (v) $\Rightarrow$ (vi).

Proof. It follows from Theorem 3.3 that (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv).

To see that (i) $\Rightarrow$ (ii), let $J \in [I]^{<\kappa}$ be given, $D_i \in [X_i]^{<\mathrm{cf}(\alpha)}$ be dense in X_i, and set $D := \Pi_{i \in J} D_i$. Clearly D is dense in $(X_J)_\kappa$, and from $\kappa << \mathrm{cf}(\alpha)$ and $|J| < \kappa$ follows $\gamma := \sup_{i \in J} |D_i| < \mathrm{cf}(\alpha)$ and then $d((X_J)_\kappa) \leq |D| \leq \gamma^{|J|} < \mathrm{cf}(\alpha)$.

We omit the argument that (iv) $\Rightarrow$ (v). This is a non-trivial result whose proof, using the hypothesis $\kappa << \mathrm{cf}(\alpha)$, is based on the "$\Delta$-system Lemma" (a.k.a. the theory of quasi-disjoint sets) due to Erdős and Rado [9], [10]. Complete detailed arguments are available in [4, 3.8], and [5, 3.25(a)], the separation hypotheses assumed there being unnecessary for the given proofs.

Now, we show that (v) implies (vi). Let W be dense in (open) $U \subseteq (X_J)_\kappa$. Since $S((X_I)_\kappa) \leq \mathrm{cf}(\alpha)$ and $\pi_J : (X_I)_\kappa \twoheadrightarrow (X_J)_\kappa$ is continuous, we have $S((X_J)_\kappa) \leq \mathrm{cf}(\alpha)$, so also $S(U) \leq$

$\mathrm{cf}(\alpha)$. Since the Souslin number is invariant in passage back and forth between a space and any of its dense subsets, we have $S(W) \leq \mathrm{cf}(\alpha)$. Then Theorem 3.3 applies to show that W is both pseudo-$(\mathrm{cf}(\alpha), \mathrm{cf}(\alpha))$-compact and pseudo-$(\alpha, \alpha)$-compact. $\square$

Remarks 3.5. (a) We draw the reader's attention to the condition in Theorem 3.4(vi) that W is assumed dense in some non-empty open subset of $(X_I)_\kappa$. So far as we are aware, this attractively weak hypothesis was first introduced into the literature relating to the extendability of continuous functions, and shown to be an adequate replacement for the more heavy-handed condition "W is dense in $(X_I)_\kappa$", by Hušek [14], [16, Theorem 8]. In our theorems in this Section and in Section 5, we continue to use this minimal hypothesis whenever it suffices.

(b) It follows from Theorem 3.4 that when X_i, κ and α are as hypothesized there, every space $(X_J)_\kappa$ with $\emptyset \neq J \in [I]^{<\kappa}$, also every set $Y \subseteq (X_I)_\kappa$ such that $\pi_J[Y] = X_J$ for all non-empty $J \in [I]^{<\kappa}$, is both pseudo-$(\mathrm{cf}(\alpha), \mathrm{cf}(\alpha))$-compact and pseudo-$(\alpha, \alpha)$-compact. We note now for later use that sometimes such specialized conclusions can be derived from hypotheses weaker than those in Theorem 3.4. We do not provide a proof of Theorem 3.6 since parts (a) and (b) are given in [5, 3.6(c)] and [5, 3.12(c)], respectively. See also [2, 3.8] for an important special case.

Theorem 3.6. *Let $\omega \leq \kappa << \mathrm{cf}(\alpha)$, $\kappa << \alpha$, $\kappa \leq \beta \leq \alpha$, $\{X_i : i \in I\}$ be a set of non-empty spaces and $Y \subseteq (X_I)_\kappa$. Assume that either*

(a) *α is regular, $\pi_J[Y] = X_J$ for every non-empty $J \in [X]^{<\kappa}$, and each $(X_J)_\kappa$ with $\emptyset \neq J \in [I]^{<\kappa}$ is pseudo-(α, β)-compact; or*

(b) *α is singular, $\pi_J[Y] = X_J$ for every non-empty $J \in [X]^{\leq \mathrm{cf}(\alpha)}$, and each $(X_J)_\kappa$ with $\emptyset \neq J \in [I]^{\leq \mathrm{cf}(\alpha)}$ is pseudo-(α, β)-compact.*

Then Y is pseudo-(α, β)-compact.

4. Functional Dependence on a Small Set of Coordinates

This Section is devoted to deriving some consequences—some new, some already accessible in [5] or [14]—of Theorem 2.2. Perhaps the simplest such result, recorded here in the interest of completeness, is the case with Z discrete.

Corollary 4.1. *Let $\omega \leq \kappa \leq \alpha$ with either $\kappa < \alpha$ or α regular, $\{X_i : i \in I\}$ be a set of non-empty spaces, Y be pseudo-(α, κ)-compact and dense in some open subset of $(X_I)_\kappa$. Let also Z be a discrete space and $f \in C(Y, Z)$. Then f depends on $< \alpha$-many coordinates.*

Proof. Apply Theorem 2.2 with O the clopen subset $O = \Delta_Z$ of $Z \times Z$. □

Remark 4.2. Although the conclusion in Corollary 4.1 is very strong, so also is the hypothesis since evidently a continuous function f from a pseudo-(α, κ)-compact space Y to a discrete space must necessarily satisfy $|f[Y]| < \alpha$. (As noted in [5, 2.4], even $|f[Y]| < \mathrm{cf}(\alpha)$ if $\alpha = \kappa$.) In particular, when $\alpha = \kappa = \omega$ the space $f[Y]$ is pseudocompact and discrete, hence is finite. The case $\alpha = \kappa = \omega$ of Corollary 4.1, asserting that every $f \in C(Y, Z)$ with Z discrete and Y pseudocompact and dense in some open subset of a space X_I must depend on finitely many coordinates, has been noted by Hušek [14, Theorem 1]. It should be remembered, however, that in general a continuous function with finite range need not depend on finitely many coordinates. Indeed Hušek [16, Example 1] has noted that for every non-trivial product X_I there are $Y \subseteq X_I$ and $f \in C(Y, \{0, 1\})$ such that f depends on no proper subset of I.

We turn next to a consideration of conditions from the literature sufficient to ensure that (certain) functions $f \in C(Y, Z)$ do depend on some "small" set of coordinates. When $Y \subseteq X_I$ and $f : Y \to Z$, we say that a set $J \subseteq I$ is *essential* (for f) if there are $x, y \in Y$ such that $d(x, y) = J$ and $f(x) \neq f(y)$; J_f denotes the set of essential indices (that is, the set of those

$i \in I$ for which $\{i\}$ is an essential set). It is not difficult to prove that

(*) if $f \in C(Y, Z)$ and there is $p \in X$ such that $\sigma(p) \subseteq Y$, then f depends on $J \subseteq I$ if and only if $J_f \subseteq J$.

In [1], we have shown that the obvious analogue of (*) in the κ-box context fails. A principal achievement of [1] is to formulate and establish the correct and useful analogue of (*) when $f \in C(Y, Z)$ with $Y \subseteq (X_I)_\kappa$. The statement of Theorem 4.5 below is straightforward, but the proof is a trifle delicate since it leans on some results from [1]. We say that an essential set J is *optimally essential* if no essential set $J' \subseteq J$ satisfies $|J'| < |J|$. As in [1], we write

$\lambda_f := \sup\{|\mathcal{J}|\colon \mathcal{J}$ is a family of pairwise disjoint optimally essential sets$\}$.

The following notation is essentially as introduced and used by Hušek in [14].

Notation 4.3. Let $\gamma \geq \omega$ and let Z be a space. Then $\gamma \in \overline{\Delta} Z$ if for every (not necessarily faithfully indexed) set $\{w(\eta) : \eta < \gamma\} \subseteq (Z \times Z) \backslash \Delta_Z$ there are $H \in [\gamma]^\gamma$ and an open neighborhood O of Δ_Z in $Z \times Z$ such that $\{w(\eta) : \eta \in H\} \cap \overline{O} = \emptyset$.

In the terminology favored by some authors (see for example [15]), a space with $\gamma \in \overline{\Delta} Z$, γ regular, is said to be a space with *weakly γ-inaccessible diagonal*.

Remark 4.4. As was noted in [14, p. 33], if γ is a regular cardinal and Z is a space with diagonal which is the intersection of $< \gamma$-many regular-closed sets then $\gamma \in \overline{\Delta} Z$. We remark that the converse of that implication fails for $\gamma = \omega$: For example, if $Z = \beta\mathbb{N}$ then $\omega \in \overline{\Delta} Z$ but ΔZ is not the intersection of finitely many regular-closed subsets of $Z \times Z$ ([15, p. 779]). Furthermore, there are consistent examples of regular Lindelöf (hence, pseudo-(ω_1, ω)-compact) spaces Z, which may be chosen even to be hereditarily Lindelöf, such that $\omega_1 \in \overline{\Delta} Z$ while Δ_Z is not a G_δ-subset of $Z \times Z$ [12]. On the other hand, under CH, every compact Hausdorff space Z, with $\omega_1 \in \overline{\Delta} Z$, is metrizable [18].

Theorem 4.5. *Let $\omega \leq \kappa \leq \alpha \leq \gamma$ with γ regular and with either $\kappa < \alpha$ or α regular, $\{X_i : i \in I\}$ be a set of non-empty spaces, Y be a pseudo-(α, κ)-compact subspace which is dense in some open subset of $(X_I)_\kappa$ and contains a dense (in Y) κ-invariant subset Y'. Let also Z be a Hausdorff space such that $\gamma \in \overline{\Delta} Z$ and $f \in C(Y, Z)$. Then f depends on $< \gamma$-many coordinates. More specifically:*

(a) *if $\kappa \leq \lambda_f$ then f depends on λ_f-many coordinates;*

(b) *if $\lambda_f < \kappa$ with κ regular then f depends on $< \kappa$-many coordinates;*

(c) *if $\lambda_f < \kappa$ with κ singular then f depends on $\leq \kappa$-many coordinates.*

Proof. We claim first that $\lambda_f < \gamma$. If that fails, then either $\lambda_f > \kappa$, or $\lambda_f = \gamma = \alpha = \kappa$, so κ is regular; in either case, according to [1, 2.30(a)], there is a family $\mathcal{J}$ of pairwise disjoint optimally essential sets such that $|\mathcal{J}| = \lambda_f$—that is, the "sup is assumed" in the definition of λ_f. Hence there is $\mathcal{J}' \subseteq \mathcal{J}$ such that $|\mathcal{J}'| = \lambda_f$, say $\mathcal{J}' = \{J_\eta : \eta < \gamma\}$.

Since J_η, for each $\eta < \gamma$, is (optimally) essential, there exists $x(\eta), y(\eta) \in Y'$ such that $d(x(\eta), y(\eta)) \subseteq J_\eta$ and $f(x(\eta)) \neq f(y(\eta))$ ([1, 2.27(a)]). Given

$$\{(f(x(\eta)), f(y(\eta))) : \eta < \gamma\} \subset (Z \times Z) \backslash \Delta_Z$$

there exists $H \in [\gamma]^\gamma$ and an open neighborhood O in $Z \times Z$ of Δ_Z such that $\{(f(x(\eta)), f(y(\eta))) : \eta \in H\} \cap \overline{O} = \emptyset$, and by Theorem 2.2 there is $J \in [I]^{<\alpha}$ such that $(f(x), f(y)) \in \overline{O}$ whenever $x, y \in Y$ satisfy $x_J = y_J$. It is clear that $J_\eta \cap J \neq \emptyset$ for all $\eta \in H$. For each $\eta \in H$, choose $i(\eta) \in J_\eta \cap J$. Then the function $\varphi : H \to J$ that sends η to $i(\eta)$ is one to one. Therefore $\gamma = |H| \leq |J| < \alpha \leq \gamma$, which is impossible. This contradiction completes the proof that $\lambda_f < \gamma$.

Now, let $\mathcal{J}''$ be a maximal family of pairwise disjoint optimally essential sets. It is not difficult to see, as in [1, 2.9], that f depends on $J := \bigcup \mathcal{J}''$. The conclusion then follows, as in [1, 2.29]. Indeed, since $|J_1| < \kappa$ for each $J_1 \in \mathcal{J}''$ ([1, 2.27(b)]), if $\kappa \leq \lambda_f$ then $|J| = \lambda_f < \gamma$; if $\lambda_f < \kappa$ with κ regular then $|J| < \kappa \leq \gamma$; and if $\lambda_f < \kappa$ with κ singular then $|J| \leq \kappa < \alpha \leq \gamma$. $\square$

If in Theorem 4.5 we take $\omega = \kappa = \alpha = \gamma$ then we obtain Theorem 2 from [14]; and when $\kappa = \omega$ and $\alpha = \gamma$ (assumed regular) we have Theorem 4 from [14].

For Theorem 4.7 we need the following simple observation (see also [14, p. 33]). As usual, when Z is a space and $A \subseteq Z$, we say that A is *regular-closed* (in Z) if there is open $U \subseteq Z$ such that $A = \overline{U}^Z$. And, Z is a *Urysohn space* if every pair of distinct points of Z are separated by disjoint regular-closed neighborhoods.

Lemma 4.6. *For each space Z, these conditions are equivalent.*

(a) *Z is a Urysohn space;*

(b) *Δ_Z is the intersection of regular-closed subspaces of $Z \times Z$.*

Proof. When (a) or (b) holds, the space Z is a Hausdorff space so Δ_Z is closed in $Z \times Z$. The desired equivalence is then immediate from the fact that if $U \times V$ and W are open in $Z \times Z$ (with $\Delta_Z \subseteq W$), then these conditions are equivalent: (a) $(U \times V) \cap W = \emptyset$; (b) $(\overline{U \times V}^{Z \times Z}) \cap W = \emptyset$; (c) $(U \times V) \cap \overline{W}^{Z \times Z} = \emptyset$. $\square$

It is clear (for γ regular) that if Z is a space such that Δ_Z is the intersection of $<\gamma$-many regular-closed subspaces of $Z \times Z$, then $\gamma \in \overline{\Delta}Z$. The assumption of this stronger hypothesis allows us in Theorem 4.7 to strengthen the conclusion of Theorem 4.5.

We do not hypothesize explicitly in Theorem 4.7 below that Z is a Hausdorff space, since Lemma 4.6 already guarantees that Z is even a Urysohn space.

Theorem 4.7. *Let $\omega \leq \kappa \leq \alpha$ with either $\kappa < \alpha$ or α regular, $\beta \geq \omega$, $\{X_i : i \in I\}$ be a set of non-empty spaces and Y be pseudo-(α, κ)-compact and dense in some open subset of $(X_I)_\kappa$.*

Let Z be a space for which Δ_Z is the intersection of β-many regular-closed subsets of $Z \times Z$. Then

(a) *each $f \in C(Y, Z)$ depends on $\leq \alpha \cdot \beta$-many coordinates; and*

(b) *if $\beta < \mathrm{cf}(\alpha)$, then each $f \in C(Y, Z)$ depends on $< \alpha$-many coordinates.*

Proof. Let $\{O_\eta : \eta < \beta\}$ be a set of open subsets of $Z \times Z$ such that $\Delta_Z = \cap\{\overline{O_\eta} : \eta < \beta\}$. For each $\eta < \beta$ there is (by Theorem 2.2) $J_\eta \in [I]^{<\alpha}$ such that $(f(x), f(y)) \in \overline{O_\eta}$ whenever $x, y \in Y$ are such that $x_{J_\eta} = y_{J_\eta}$. We set $J = \cup\{J_\eta : \eta < \beta\}$. Then f depends on J. Clearly $|J| \leq \alpha \cdot \beta$, and $|J| < \alpha$ if $\beta < \mathrm{cf}(\alpha)$. □

Remark 4.8. The case of Theorem 4.7 with κ arbitrary and with Z metrizable (hence, $\beta = \omega$) is given in [5, 10.2]. The case of Theorem 4.7 with $\kappa = \omega$ and $\beta < \alpha$ is due to Hušek [14, Theorem 3] (see also [16, Theorem 5]). For other criteria sufficient to ensure when $\kappa = \omega$ that every $f \in C(Y, Z)$ depends on $< \alpha$-many coordinates, see [13, Theorems 1 and 2].

5. Extending Continuous Functions

Discussion 5.1. (a) As indicated in our Abstract, the focus of this paper is on the question of extending functions of the form $f \in C(Y, Z)$ to functions of the form $\overline{f} \in C(Y', Z')$ with $Y \subseteq Y'$ and $Z \subseteq Z'$. Our methods and our results are intelligible only for spaces Y for which $Y \subseteq X_I$ or $Y \subseteq (X_I)_\kappa$, typically with Y dense in $(X_I)_\kappa$. In every theorem, the proof proceeds in two steps. First, f is shown to depend on some set of coordinates. Second, it is shown that every function which so depends must extend continuously over its closure in $(X_I)_\kappa$. Step 1 is based on Theorem 2.2 and its various consequences given in Section 4. In each of our principal theorems, Step 2 is based on Lemma 5.2 below. This result appears as Lemma 10.3 in [5] (though with the unnecessary global assumption made in [5] of complete regularity); it also appears, without such assumptions, in [2, 3.2]. To keep the present treatment relatively self-contained, we include a brief proof here as given in [2]. For additional and extended details see [5, 10.3] and its proof, which does not depend on [5, 10.1].

(b) We paraphrase slightly our commentary from [2].

Lemma 5.2 is a simple result. We observe a qualitative distinction in flavor between Lemma 5.2 and those deeper theorems of General Topology which, for Y dense in some space Y' and for some spaces $Z \subseteq Z'$, guarantee that every $f \in C(Y, Z)$ extends to $\overline{f} \in C(Y', Z')$ (e.g., the extension theorem of Stone and Čech, Lavrentieff's theorem, and so forth). Typically there, $Z' = \text{range}(\overline{f})$ properly contains $Z = \text{range}(f)$, and $\overline{f}$ is defined at the points of $Y' \backslash Y$ using some sort of completeness property, or an argument of Baire Category type, for Z'. In Lemma 5.2, in contrast, the hypotheses on the disposition or placement of $Y \subseteq (X_I)_\kappa$ are sufficiently strong that each point $p \in X = X_I$ associates naturally with a point $y \in Y$ such that $p_J = y_J$ for some non-empty $J \subset I$, and the natural definition $\overline{f}(p) = f(y)$ renders unnecessary the consideration of any completeness properties which Z may enjoy, and ensures also that $\text{range}(\overline{f}) = \text{range}(f)$. To reiterate explicitly: In the theorems in this paper, the domain properly increases in passing from f to $\overline{f}$, but the range does not.

(c) The reader will notice that in the extension theorems we prove below, there are hypotheses both on the domain space $Y \subseteq (X_I)_\kappa$ and on the range space Z. These sets of hypotheses are necessarily related and co-dependent. To see this in a stark manner, suppose that the spaces X_i are Hausdorff spaces and consider the a-typical case $Y = Z$ with Y dense in $(X_I)_\kappa$. The space $Y = Z$ can satisfy simultaneously our hypotheses on Y and on Z only in the extreme case $Y = Z = (X_I)_\kappa$. Indeed otherwise the continuous extension $\overline{f}$ of the identity function $f = \text{id}_Y : Y \twoheadrightarrow Y = Z$ will retract $(X_I)_\kappa$ onto its proper dense subspace $Y = Z$, which is impossible (when $(X_I)_\kappa$ is a Hausdorff space).

We discuss further the relation between the condition on the space Y, that $\pi_J[Y] = X_J$ for every suitably small $J \subseteq I$, and those conditions on Z which help to ensure that each function $f \in C(Y, Z)$ depends on some small set of coordinates (see Theorems 4.5 and 4.7). To appreciate the co-dependence of these conditions, consider again the a-typical case $Y = Z$ with the identity function $f := \mathrm{id}_Y : Y \twoheadrightarrow Y = Z$. It is clear that the conditions

(i) f depends on some $J \subseteq I$ such that $\omega \leq |J| < |I|$ and
(ii) $\pi_{J'}[Y] = X_{J'}$ for each $J' \in [I]^{\leq |J|}$

are incompatible. Indeed if (i) holds and $i \in I \backslash J$ is chosen so that $|X_i| > 1$, then with $J' := J \cup \{i\}$ there are distinct $p, q \in X_{J'}$ such that $p_J = q_J$; then (ii) fails, for by (i) there are no $x, y \in Y$ such that $x_{J'} = p$ and $y_{J'} = q$.

Lemma 5.2. *Let $\omega \leq \kappa \leq \alpha$, $\{X_i : i \in I\}$ be a set of non-empty spaces, Y be a subspace of $(X_I)_\kappa$ such that $\pi_J[Y] = X_J$ for every non-empty $J \in [I]^{<\alpha}$, and Z be a space. Then every $f \in C(Y, Z)$ which depends on $< \alpha$-many coordinates extends to a continuous function $\overline{f} : (X_I)_\kappa \to Z$.*

Proof. Let f depend on $\emptyset \neq J \in [I]^{<\alpha}$. For $p \in X_I$ choose $y \in Y$ such that $p_J = y_J$, and define $\overline{f} : X_I \to Z$ by $\overline{f}(p) = f(y)$. Since $\pi_{J'}[U] = \pi_{J'}[U \cap Y]$ for each basic open set U of $(X_I)_\kappa$ and for each set $J' \in [I]^{<\alpha}$, the continuity of $\overline{f}$ on $(X_I)_\kappa$ follows from the continuity of f on Y. □

All the results which follow are essentially consequences of juxtaposing Theorem 4.5 or Theorem 4.7 and Lemma 5.2 with other applicable data.

Consequences of Theorem 4.5 and Lemma 5.2

Theorem 5.3. *Let $\omega \leq \kappa \leq \alpha \leq \gamma$ with γ regular and with either $\kappa < \alpha$ or α regular, $\{X_i : i \in I\}$ be a set of non-empty spaces, Y be a pseudo-(α, κ)-compact subspace of $(X_I)_\kappa$ which contains a dense (in Y) κ-invariant subset and such that $\pi_J[Y] = X_J$ for every non-empty $J \in [I]^{<\gamma}$. Let also Z be a*

Hausdorff space such that $\gamma \in \overline{\Delta} Z$. Then every $f \in C(Y, Z)$ extends to a continuous function $\overline{f} : (X_I)_\kappa \to Z$.

Proof. It follows from Theorem 4.5 that f depends on $< \gamma$-many coordinates, so Lemma 5.2 applies (with γ replacing α there). □

Remark 5.4. Clearly the two hypotheses Y contains a dense κ-invariant subset and $\pi_J[Y] = X_J$ in Theorem 5.3 could be replaced by the single stronger hypothesis that $\Sigma_\gamma(p) \subseteq Y$, for some $p \in X_I$.

Theorem 5.5. *Let $\omega \leq \kappa << \alpha \leq \gamma$ with γ regular, $\kappa <<$ cf(α), $\{X_i : i \in I\}$ be a set of non-empty spaces, Y be a subspace of $(X_I)_\kappa$ which contains a dense (in Y) κ-invariant subset and is such that $\pi_J[Y] = X_J$ for every non-empty $J \in [I]^{<\gamma}$. Assume also that either*

(a) *α is regular, and $(X_J)_\kappa$ is pseudo-(α, κ)-compact for every non-empty $J \in [I]^{<\kappa}$; or*

(b) *α is singular, and $(X_J)_\kappa$ is pseudo-(α, κ)-compact for every non-empty $J \in [I]^{\leq \mathrm{cf}(\alpha)}$; or*

(c) *$S((X_I)_\kappa) \leq$ cf(α).*

Let Z be a Hausdorff space such that $\gamma \in \overline{\Delta} Z$. Then every $f \in C(Y, Z)$ extends to a continuous function $\overline{f} : (X_I)_\kappa \to Z$.

Proof. It follows in (a) and (b) from Theorem 3.6 that Y itself is pseudo-(α, κ)-compact. Thus the claims in (a) and (b) follow from Theorem 5.3.

It follows in (c) from Theorem 3.4(b), since Y is dense in $(X_I)_\kappa$, that Y is pseudo-(α, α)-compact. Thus, the claim in (c) follows from Theorem 5.3. □

Consequences of Theorem 4.7 and Lemma 5.2

Theorem 5.6. *Let $\omega \leq \kappa \leq \alpha$ with either $\kappa < \alpha$ or α regular, $\beta \geq \omega$, $\{X_i : i \in I\}$ be a set of non-empty spaces, and Y be a pseudo-(α, κ)-compact subspace of $(X_I)_\kappa$. Assume also that either*

(a) $\pi_J[Y] = X_J$ *for every non-empty* $J \in [I]^{\leq \alpha\cdot\beta}$*; or*

(b) $\beta < \mathrm{cf}(\alpha)$*, and* $\pi_J[Y] = X_J$ *for every non-empty* $J \in [I]^{<\alpha}$.

Let Z *be a space for which* Δ_Z *is the intersection of* β*-many regular-closed subsets of* $Z \times Z$*. Then every* $f \in C(Y, Z)$ *extends to a continuous function* $\overline{f} : (X_I)_\kappa \to Z$*.*

Proof. According to Theorem 4.7, each such f depends on $\leq \alpha \cdot \beta$-many coordinates in (a), and each such f depends on $< \alpha$-many coordinates in (b). Lemma 5.2 then applies (with α replaced by $(\alpha \cdot \beta)^+$ in (a)). □

The special case $\kappa = \omega = \beta$ of the above theorem is Theorem 3.2 in [22]. Theorem 5.6 has this consequence.

Theorem 5.7. *Let* $\omega \leq \kappa << \alpha$*,* $\kappa << \mathrm{cf}(\alpha)$*,* $\beta \geq \omega$*,* $\{X_i : i \in I\}$ *be a set of non-empty spaces, and* Y *be a subspace of* $(X_I)_\kappa$*. Assume also that either* $S((X_I)_\kappa) \leq \mathrm{cf}(\alpha)$*, or that* α *is regular and* $\pi_J[Y]$ *is pseudo-*(α, κ)*-compact for every non-empty* $J \in [I]^{<\kappa}$*, or that* α *is singular and* $\pi_J[Y]$ *is pseudo-*(α, κ)*-compact for every non-empty* $J \in [I]^{\leq \mathrm{cf}(\alpha)}$*. Assume further that either*

(a) $\pi_J[Y] = X_J$ *for every non-empty* $J \in [I]^{\leq \alpha\cdot\beta}$*; or*

(b) $\beta < \mathrm{cf}(\alpha)$ *and* $\pi_J[Y] = X_J$ *for every non-empty* $J \in [I]^{<\alpha}$.

Let Z *be a space for which* Δ_Z *is the intersection of* β*-many regular-closed subsets of* $Z \times Z$*. Then every* $f \in C(Y, Z)$ *extends to a continuous function* $\overline{f} : (X_I)_\kappa \to Z$*.*

Proof. If $S((X_I)_\kappa) \leq \mathrm{cf}(\alpha)$ then it follows from Theorem 3.4, since Y is dense in $(X_I)_\kappa$, that Y is pseudo-(α, α)-compact. Thus, the claim follows from Theorem 5.6.

In the other two cases it follows from Theorem 3.6 that Y is pseudo-(α, κ)-compact, so again the claim follows from Theorem 5.6. □

When in Theorem 5.7 Z is a metric space (hence $\beta = \omega$) we obtain [5, 10.4(i)], and together with Theorem 3.4(i) and (iv) with $\kappa = \omega$ we get [5, 10.5].

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