ON PATH COVERINGS OF HYPERCUBES WITH ONE FAULTY VERTEX

N. CASTAÑEDA, V. GOCHEV, I. GOTCHEV, AND F. LATOUR

Abstract. Let \( n \geq 5 \) and \( Q_n \) be the \( n \)-dimensional binary hypercube. Also, let \( u \) be a faulty even (odd) vertex in \( Q_n \), \( u_1 \) and \( u_2 \) be distinct even (odd) vertices of \( Q_n - \{u\} \) and \( v_1, v_2, v_3 \) and \( v_4 \) be distinct odd (even) vertices of \( Q_n - \{u\} \). We prove that there exist three paths in \( Q_n - \{u\} \), one joining \( u_1 \) and \( u_2 \), one joining \( v_1 \) and \( v_2 \) and one joining \( v_3 \) and \( v_4 \), such that every vertex of \( Q_n - \{u\} \) belongs to exactly one of the paths. Also, under the same hypotheses, there exist three paths in \( Q_n - \{u\} \), one joining \( u_1 \) and \( v_1 \), one joining \( u_2 \) and \( v_2 \) and one joining \( v_3 \) and \( v_4 \), such that every vertex of \( Q_n - \{u\} \) belongs to exactly one of the paths.

1. Introduction

A path covering of a graph \( G = (V, E) \) is a set of vertex-disjoint simple paths in \( G \) with the property that every vertex of \( G \) is contained in exactly one of the paths. For \( k \geq 1 \), a \( k \)-path covering of \( G \) is a path covering consisting of \( k \) paths. In this paper we are interested in 3−path coverings of induced subgraphs of the \( n \)-dimensional binary hypercube \( Q_n \). An induced subgraph of \( Q_n \) is also known as a faulty hypercube.

The vertices of \( Q_n \) are binary vectors of length \( n \). A vertex is odd if it contains an odd number of 1s and even if it contains an even number of 1s. In order to make our results as general as possible, the vertices of one parity will be called green and those of the other parity will be called red.

One of the most basic theorems on path coverings of \( Q_n \) is Havel’s Lemma [1], which states that given any two vertices of opposite parity in \( Q_n \), with \( n \geq 1 \), there exists a Hamiltonian path with these two vertices as endpoints, or in the terminology of [2], \( Q_n \), for each \( n \geq 1 \).
is Hamilton laceable. For $2$–path coverings of $Q_n$ the following two results are known.

**Lemma 1.1** ([3]). Let $n \geq 2$, $g_1, g_2 \in V(Q_n)$ be distinct green vertices and $r_1, r_2 \in V(Q_n)$ be distinct red vertices. Then there exists a $2$–path covering of $Q_n$ with one path joining $g_1$ and $r_1$ and another path joining $g_2$ and $r_2$.

**Lemma 1.2** ([4], [5]). Let $n \geq 4$, $g_1, g_2 \in V(Q_n)$ be distinct green vertices and $r_1, r_2 \in V(Q_n)$ be distinct red vertices. Then there exists a $2$–path covering of $Q_n$ with one path joining $g_1$ and $r_1$ and another joining $g_2$ and $r_2$. The claim is not true when $2 \leq n \leq 3$.

For $3$–path coverings of $Q_n$ the following result was obtained in [5].

**Lemma 1.3** ([5]). Let $n \geq 5$, $g_1, g_2, g_3 \in V(Q_n)$ be distinct green vertices and $r_1, r_2, r_3 \in V(Q_n)$ be distinct red vertices. Then there exists a $3$–path covering of $Q_n$ with one path joining $g_1$ and $r_1$, another path joining $g_2$ and $r_2$, and another path joining $g_3$ and $r_3$. The claim is not true when $3 \leq n \leq 4$.

Another basic result on path coverings of $Q_n$ with one deleted vertex is the following.

**Lemma 1.4** ([6]). Let $n \geq 2$, $g \in V(Q_n)$, and $r_1, r_2 \in V(Q_n)$ be distinct red vertices. Then there exist a Hamiltonian path for $Q_n - \{g\}$ joining $g_1$ and $g_2$.

In the terminology of [6] the above lemma states that $Q_n$ is hyper-Hamilton laceable.

For $2$–path coverings of $Q_n$ with one deleted vertex the following result is proved in [5].

**Lemma 1.5** ([5]). Let $n \geq 4$, $g, g_1 \in V(Q_n)$ be distinct green vertices and $r_1, r_2, r_3 \in V(Q_n)$ be distinct red vertices. Then there exists a $2$–path covering of $Q_n - \{g\}$ with one path joining $g_1$ and $r_1$ and another joining $r_2$ and $r_3$. The claim is not true when $2 \leq n \leq 3$.

Castañeda and Gotchev continued the study of path coverings of induced subgraphs of $Q_n$ in [5] where they considered the following more general problem:

**Problem 1.6.** Let $M, C, N$ and $O$ be nonnegative integers and $F$ be a subset of $V(Q_n)$ such that the cardinality of $F$ is $M$ and the absolute value of the difference between the number of odd vertices in $F$ and the number of even vertices in $F$ is $C$. Let also $a_1, \ldots, a_{N+O}$ and
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Let \( b_1, \ldots, b_{N+O} \) be distinct vertices in \( V(Q_n) \setminus F \), such that there are \( N \) pairs \( a_i, b_i \) of opposite parities and \( O \) pairs \( a_i, b_i \) of matching parities. Find the smallest \( m \) such that if \( n \geq m \) then, regardless of the choice of \( F \) and \( a_1, \ldots, a_{N+O}, b_1, \ldots, b_{N+O} \), there exists an \((N+O)\)-path covering of the subgraph induced by \( V(Q_n) \setminus F \) with one path joining \( a_i \) and \( b_i \), for each \( i \in \{1, \ldots, N+O\} \).

In [5], the smallest \( m \) defined in Problem 1.6 is denoted \([M, C, N, O]\). For example, using this notation, Havel’s lemma states \([0, 0, 1, 0] = 1\), Lemma 1.1 states \([0, 0, 2, 0] = 2\), Lemma 1.2 states \([0, 0, 0, 2] = 4\), Lemma 1.3 states \([0, 0, 3, 0] = 5\), Lemma 1.4 states \([1, 1, 0, 1] = 2\) and Lemma 1.5 states \([1, 1, 1, 1] = 4\).

In the following table are given some of the values of \([M, C, N, O]\) known to us. Most of them were obtained in [5]. [7] contains \([0, 0, 1, 2] = 4\) and [8] contains \([4, 0, 2, 0] = 5\) and \([7, 1, 0, 1] = 6\) (the latter result is not included in the table). The rows represent admissible combinations of \( M \) and \( C \) and the columns contain all the values of \( N \) and \( O \) such that \( N + O \leq 3 \). Each star in the table represents an impossible case. The missing entries in the table correspond to values of \([M, C, N, O]\) that we do not know yet. The inequalities in the table represent an upper or lower bound of the corresponding entry.

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In this paper we continue to investigate the existence of path coverings with prescribed ends in hypercubes \( Q_n \) with one faulty vertex. More specifically, we prove the following two results: Let \( u \) be a faulty even (odd) vertex in \( Q_n \), \( u_1 \) and \( u_2 \) be distinct even (odd) vertices of \( Q_n \setminus \{u\} \) and \( v_1, v_2, v_3 \) and \( v_4 \) be distinct odd (even) vertices of \( Q_n \setminus \{u\} \). If \( n \geq 5 \), then there exist a 3-path covering of \( Q_n \setminus \{u\} \), with one path joining \( u_1 \) and \( u_2 \), one path joining \( v_1 \) and \( v_2 \) and one path joining \( v_3 \) and \( v_4 \). Also, under the same hypotheses, there exist a 3-path covering...
of $Q_n - \{u\}$, one joining $u_1$ and $v_1$, one joining $u_2$ and $v_2$ and one joining $v_3$ and $v_4$. Both claims are not always true when $n \leq 4$. In the proofs we use many of the results contained in the above table. In the notation of [5], our results are equivalent to the statements $[1,1,2,1] = 5$ and $[1,1,0,3] = 5$ (these values are marked with a dagger in the above table).

2. Definitions and notation

In the proofs that follow we shall use the notation already introduced in [5].

In many cases, it will be useful to view $Q_n$ as two copies of $Q_{n-1}$, denoted $Q_n^{\text{top}}$ and $Q_n^{\text{bot}}$, with the vertices of $Q_n^{\text{top}}$ and those of $Q_n^{\text{bot}}$ joined by edges in pairs. $Q_n^{\text{top}}$ and $Q_n^{\text{bot}}$ will be called the top plate and the bottom plate, respectively. Without loss of generality, we may assume that the vertices of $Q_n^{\text{top}}$ all share the same $v$-coordinate (say $v = 1$), and the vertices of $Q_n^{\text{bot}}$ all share the same $v$-coordinate (say $v = 0$). The coordinates other than the $v$-coordinate will be denoted by the letters $x,y,z,...$.

Any path in $Q_n$ with an endpoint $r$ can be written uniquely as $(r,\eta)$, where $\eta$ is a sequence of letters in $\{x,y,z,v,...\}$ (such a sequence will be called a word). For example, in $Q_5$, denote the five coordinates by $x,y,z,w,v$ in that order. Then the path starting at $r = (0,1,0,0,1)$, going through $(1,1,0,0,1)$, $(1,1,0,0,0)$, $(1,0,0,0,0)$ and ending at $(1,0,1,0,0)$ can be written $(r,xvyz)$.

The initial vertex of the path $(r,\eta)$ is $r$; if the terminal vertex is $s$, then the same path will sometimes be written $(r,\eta;s)$.

We shall also use the following notation: $\omega^R$ means the reverse word of $\omega$; $\omega'$ denotes the word obtained after the last letter is deleted from $\omega$; $\omega^*$ is the word obtained after the first letter is deleted from $\omega$ and $\varphi(\omega)$ is the first letter of $\omega$.

3. $[1,1,0,3] = 5$

**Theorem 3.1** ($[1,1,0,3] = 5$). Let $n \geq 5$, and $r,r_1,r_2,g_1,g_2,g_3,g_4$ be three distinct red and four distinct green vertices in $Q_n$. Then there exists a 3-path covering $(r_1,\xi;r_2)$, $(g_1,\eta;g_2)$, $(g_3,\zeta;g_4)$ of $Q_n - \{r\}$. The claim is not always true if $n < 5$.

**Proof.** The following counterexample shows that $[1,1,0,3] \geq 5$. 


Let \( n = 4 \) and \( r = (0, 1, 1, 0) \). Let also \( r_1 = (0, 0, 1, 1), r_2 = (0, 1, 0, 1), g_1 = (1, 0, 1, 1), g_2 = (1, 1, 1, 0), g_3 = (1, 1, 0, 1), \) and \( g_4 = (1, 0, 0, 0) \) be vertices in \( Q_4 - \{ r \} \). Then one can directly verify that a 3-path covering of \( Q_4 - \{ r \} \) with paths connecting \( r_1 \) to \( r_2, g_1 \) to \( g_2, \) and \( g_3 \) to \( g_4 \) does not exist.

Now, let \( n \geq 5 \). Let \( v \) be a coordinate where \( r_1 \) and \( r_2 \) do not agree, and use that coordinate to split \( Q_n \) into two plates. Without loss of generality, we can assume that \( r \) and \( r_1 \) are on the top plate and \( r_2 \) is on the bottom plate. There are six different cases to consider that depend on the distribution of the green terminals on the plates.

**Case (1)** \( g_1, g_2, g_3, g_4 \) are on the top plate.

Use \([2, 2, 0, 2] = 4\) to find a 2-path covering \( \gamma_1, \gamma_2 \) of \( Q_n^{\text{top}} - \{ r, r_1 \} \) that connects \( g_1 \) to \( g_2 \) and \( g_3 \) to \( g_4 \), respectively. Use \([0, 0, 1, 0] = 1\) to find a Hamiltonian path \( \gamma_3 \) of \( Q_n^{\text{bot}} \) that connects \( r_1v \) to \( r_2 \). Then \( \gamma_1, \gamma_2, (r_1, v\gamma_3) \) is the required 3-path covering of \( Q_n - \{ r \} \).

**Case (2)** \( g_1, g_2, g_4 \) are on the top plate and \( g_3 \) is on the bottom plate.

We consider two cases: \( r_1v \neq g_4 \) and \( r_1v = g_4 \).

Assume that \( r_1v \neq g_4 \). Choose a green vertex \( g \) in \( Q_n^{\text{top}} - \{ g_1, g_2, g_3 \} \) such that \( gv \neq r_2 \). Use \([2, 2, 0, 2] = 4\) to find a 2-path covering \( \gamma_1, \gamma_2 \) of \( Q_n^{\text{top}} - \{ r, r_1 \} \) that connects \( g_1 \) to \( g_2 \) and \( g_3 \) to \( g \), respectively. Use \([0, 0, 2, 0] = 2\) to find a 2-path covering \( \gamma_3, \gamma_4 \) of \( Q_n^{\text{bot}} \) that connects \( r_1v \) to \( r_2 \) and \( gv \) to \( g_4 \), respectively. Then \( \gamma_1, (g_3, \gamma_2\gamma_4), (r_1, v\gamma_3) \) is the required 3-path covering of \( Q_n - \{ r \} \).

Assume now that \( r_1v = g_4 \). Let \( g \) be a green vertex in \( Q_n^{\text{top}} - \{ g_1, g_2, g_3 \} \) adjacent to \( r_1 \). Such vertex exists since \( n \geq 5 \). Let \( g = r_1x \) for some letter \( x \). There are two possibilities: \( gv = r_2 \) and \( gv \neq r_2 \).

Assume that \( gv = r_2 \). Then \( g_3v \neq r_2 \). Use \([2, 2, 0, 2] = 4\) to find a 2-path covering \( \gamma_1, \eta \) of \( Q_n^{\text{top}} - \{ r, r_1 \} \) that connects \( g_1 \) to \( g_2 \) and \( g \) to \( g_3 \), respectively. Use \([0, 0, 2, 0] = 2\) to find a 2-path covering \( \gamma_2, \gamma_3 \) of \( Q_n^{\text{bot}} \) that connects \( g\eta'v \) to \( r_2 \) and \( g_3v \) to \( g_4 \), respectively. Then \( \gamma_1, (g_3, \gamma_2\gamma_3), (r_1, x\eta'v\gamma_3) \) is the required 3-path covering of \( Q_n - \{ r \} \).

Finally, assume that \( gv \neq r_2 \). Use \([2, 2, 0, 2] = 4\) to find a 2-path covering \( \gamma_1, \eta \) of \( Q_n^{\text{top}} - \{ r, r_1 \} \) that connects \( g_1 \) to \( g_2 \) and \( g_3 \) to \( g \), respectively. Use \([0, 0, 0, 2] = 4\) to find a 2-path covering \( \gamma_2, \gamma_3 \) of \( Q_n^{\text{bot}} \) that connects \( g_3\eta'v \) to \( g_4 \) and \( gv \) to \( r_2 \), respectively. Then \( \gamma_1, (g_3, \eta'v\gamma_2), (r_1, xv\gamma_3) \) is the required 3-path covering of \( Q_n - \{ r \} \).

**Case (3)** \( g_1, g_2 \) are on the top plate and \( g_3, g_4 \) are on the bottom plate.
Choose a green vertex \( g \) in \( Q_n^{\text{top}} \setminus \{g_1, g_2, r_2v\} \). Use \([1, 1, 1, 1] = 4\) to find a 2–path covering \( \gamma_1, \gamma_2 \) of \( Q_n^{\text{top}} \setminus \{r\} \) that connects \( g_1 \) to \( g_2 \) and \( r_1 \) to \( g \), respectively. Use \([0, 0, 0, 2] = 4\) to find a 2–path covering \( \gamma_3, \gamma_4 \) of \( Q_n^{\text{bot}} \) that connects \( g \) \( v \) to \( r_2 \) and \( g_3 \) to \( g_4 \), respectively. Then \( \gamma_1, (r_1, \gamma_2 v \gamma_3), \gamma_4 \) is the required 3–path covering of \( Q_n \setminus \{r\} \).

**Case (4)** \( g_1, g_3 \) are on the top plate and \( g_2, g_4 \) are on the bottom plate.

Without loss of generality we can assume that \( g_1 v \neq r_2 \) for if not then \( g_3 v \neq r_2 \). Use \([2, 0, 1, 0] = 4\) to find a Hamiltonian path \( \gamma \) of \( Q_n^{\text{top}} \{r, g_1\} \) that connects \( r_1 \) to \( g_3 \). Since \( n \geq 5 \), the length of this path is at least 13, and so there are at least 7 edges \((\hat{r}, \hat{g})\) in \( \gamma \), where \( \hat{r} \) is red, \( \hat{g} \) is green and \( \hat{r} \) is closer to \( r_1 \) along \( \gamma \) than \( \hat{g} \) is. Choose one such edge \((\hat{r}, \hat{g})\) with the property that \( \hat{r} \hat{v} \notin \{g_2, g_4\} \) and \( \hat{g} \hat{v} \neq r_2 \). Let \( x \) be such that \( \hat{g} = \hat{r} x \) and write \( \gamma = \eta x \theta \), with \( \hat{r} = r_1 \eta, g_3 = \hat{g} \theta \). Use \([0, 0, 1, 2] = 4\) to find a 3–path covering \( \gamma_1, \gamma_2, \gamma_3 \) of \( Q_n^{\text{bot}} \) that connects \( \hat{r} \hat{v} \) to \( r_2 \), \( \hat{g} \hat{v} \) to \( g_4 \), and \( g_1 v \) to \( g_2 \), respectively. Then \( \gamma_1, (r_1, \gamma_2 v \gamma_3), \gamma_5 \) is the required 3–path covering of \( Q_n \setminus \{r\} \).

**Case (5)** \( g_1 \) is on the top plate and \( g_2, g_3, g_4 \) are on the bottom plate.

Choose green vertices \( g \) and \( \hat{g} \) in \( Q_n^{\text{top}} \{g_1\} \) such that \( g v \neq r_2 \) and \( \hat{g} \hat{v} \neq r_2 \). Use \([1, 1, 1, 1] = 4\) to find a 2–path covering \( \gamma_1, \gamma_2 \) of \( Q_n^{\text{top}} \{r\} \) that connects \( g_1 \) to \( \hat{g} \) and \( r_1 \) to \( g \), respectively. Use \([0, 0, 1, 2] = 4\) to find a 3–path covering \( \gamma_3, \gamma_4, \gamma_5 \) of \( Q_n^{\text{bot}} \) that connects \( g v \) to \( r_2 \), \( \hat{g} \hat{v} \) to \( g_2 \), and \( g_3 \) to \( g_4 \), respectively. Then \( \gamma_1, (g_1, \gamma_1 v \gamma_4), (r_1, \gamma_2 v \gamma_3), \gamma_5 \) is the required 3–path covering of \( Q_n \setminus \{r\} \).

**Case (6)** \( g_1, g_2, g_3, g_4 \) are on the bottom plate.

Use \([1, 1, 0, 1] = 2\) to find a Hamiltonian path \( \gamma \) of \( Q_n^{\text{bot}} \{r_2\} \) that connects \( g_1 \) to \( g_2 \). We can assume that \( \gamma = \eta \theta \zeta \) with \( g_3 = g_1 \eta, g_4 = g_2 \theta \), by renumbering \( g_3 \) and \( g_4 \), if necessary. Use \([1, 1, 1, 1] = 4\) to find a 2–path covering \( \gamma_1, \gamma_2 \) of \( Q_n^{\text{top}} \{r\} \) that connects \( g_1 \eta' v \) to \( g_4 \phi' (\zeta) v \) and \( r_2 v \) to \( r_1 \), respectively. Then \( \theta, (r_2, \nu \gamma_2), (g_1, \eta' v \gamma_1 v \zeta' \phi) \) is the required 3–path covering of \( Q_n \setminus \{r\} \).

\( \square \)

4. \([1, 1, 2, 1] = 5\)

**Theorem 4.1** ([1, 1, 2, 1] = 5). Let \( n \geq 5 \) and \( r, r_1, r_2, g_1, g_2, g_3, g_4 \) be three distinct red and four distinct green vertices in \( Q_n \). Then there exists a 3–path covering \( (g_1, \xi; r_1), (g_2, \eta; r_2), (g_3, \zeta; g_4) \) of \( Q_n \setminus \{r\} \).

The claim is not true if \( n < 5 \).

**Proof.** The following counterexample shows that \([1, 1, 2, 1] \geq 5\).
Let \( n = 4 \) and \( r = (0, 1, 1, 0) \). Let also \( r_1 = (0, 0, 1, 1), r_2 = (0, 0, 0, 0), g_1 = (0, 1, 0, 0), g_2 = (0, 1, 1, 1), g_3 = (0, 0, 1, 0), \) and \( g_4 = (0, 0, 0, 1) \) be vertices in \( Q_4 - \{r\} \). Then one can directly verify that a 3-path covering of \( Q_4 - \{r\} \) with paths connecting \( g_1 \) to \( r_1 \), \( g_2 \) to \( r_2 \), and \( g_3 \) to \( g_4 \) does not exist.

Now, let \( n \geq 5 \). Let \( v \) be a coordinate where \( r_1 \) and \( r_2 \) do not agree, and use that coordinate to split \( Q_n \) into two plates. Without loss of generality, we can assume that \( r \) and \( r_1 \) are on the top plate and \( r_2 \) is on the bottom plate. There are sixteen different cases to consider that depend on the distribution of the green terminals on the plates.

**Case (1)*** \( g_1, g_2, g_3, g_4 \) are on the top plate.

Use \([2, 2, 0, 2] = 4\) to find a 2-path covering \((g_1, \gamma_1; g_2), (g_3, \gamma_2; g_4)\) of \( Q_4^{\text{top}} - \{r, r_1\} \). Either \( g_1 v \neq r_2 \) or \( g_2 v \neq r_2 \). If \( g_1 v \neq r_2 \) then use \([0, 0, 2, 0] = 2\) to find a 2-path covering \((g_1 v, \gamma_3; r_1 v), (g_1 \varphi(\gamma_1) v, \gamma_4; r_2)\) of \( Q_4^{\text{bot}} \). Then \((g_1, v \gamma_3 v; r_1), (g_2, (\gamma_1^R \gamma_3) v \gamma_4; r_2), (g_3, \gamma_2; g_4)\) is the required 3-path covering of \( Q_n - \{r\} \). If \( g_2 v \neq r_2 \) then use \([0, 0, 0, 2] = 4\) to find a 2-path covering \((g_2 v, \gamma_3; r_2), (g_1 \gamma_1 v, \gamma_4; r_1 v)\) of \( Q_4^{\text{bot}} \). Then \((g_1, \gamma_1 v \gamma_3 v; r_1), (g_2, v \gamma_3; r_2), (g_3, \gamma_2; g_4)\) is the required 3-path covering of \( Q_n - \{r\} \).

**Case (2)*** \( g_2, g_3, g_4 \) are on the top plate and \( g_1 \) is on the bottom plate.

Use \([3, 1, 0, 1] = 4\) to find a Hamiltonian path \((g_3, \gamma_1; g_4)\) of \( Q_4^{\text{top}} - \{r, r_1, g_2\} \).

If \( g_2 v \neq r_2 \) and \( r_1 v \neq g_1 \) use \([0, 0, 0, 2] = 4\) to find a 2-path covering \((g_1, \gamma_2; r_1 v), (g_2, \gamma_3; r_2)\) of \( Q_4^{\text{bot}} \). Then \((g_1, \gamma_2 v; r_1), (g_2, v \gamma_3; r_2), (g_3, \gamma_1; g_4)\) is the required 3-path covering of \( Q_n - \{r\} \).

If \( g_2 = r_2 \) and \( r_1 v \neq g_1 \) use \([1, 1, 0, 1] = 2\) to find a Hamiltonian path \((g_1, \gamma_2; r_1 v)\) of \( Q_4^{\text{bot}} - \{r_2\} \). Then \((g_1, \gamma_2 v; r_1), (g_2, v; r_2), (g_3, \gamma_1; g_4)\) is the required 3-path covering of \( Q_n - \{r\} \).

If \( g_2 v \neq r_2 \) and \( r_1 v = g_1 \) use \([1, 1, 0, 1] = 2\) to find a Hamiltonian path \((g_2 v, \gamma_2; r_2)\) of \( Q_4^{\text{bot}} - \{g_1\} \). Then \((g_1, v; r_1), (g_2, v \gamma_2; r_2), (g_3, \gamma_1; g_4)\) is the required 3-path covering of \( Q_n - \{r\} \).

Finally, suppose \( g_2 v = r_2 \) and \( r_1 v = g_1 \). Let \( x, y \) be the first two letters of \( \gamma \) and write \( \gamma_1 = x y \eta \). Then \( g_3 x v \neq g_1 \) and \( g_3 x y v \neq r_2 \), so we can use \([2, 0, 1, 0] = 4\) to find a Hamiltonian path \((g_3 x v, \gamma_2; g_3 x y v)\) of \( Q_4^{\text{bot}} - \{g_1, r_2\} \). Then \((g_1, v; r_1), (g_2, v; r_2), (g_3, x v \gamma_2 v \eta; g_4)\) is the required 3-path covering of \( Q_n - \{r\} \).

**Case (3)*** \( g_1, g_3, g_4 \) are on the top plate and \( g_2 \) is on the bottom plate.
Use \([1,1,1,1] = 4\) to find a 2-path covering \((g_1, \gamma_1; r_1), (g_3, \gamma_2; g_4)\) of \(Q_n^{top} - \{r\}\) and use \([0,0,1,0] = 1\) to find a Hamiltonian path \((g_2, \gamma_3; r_2)\) of \(Q_n^{bot}\). Then \((g_1, \gamma_1; r_1), (g_2, \gamma_3; r_2), (g_3, \gamma_2; g_4)\) is the required 3-path covering of \(Q_n - \{r\}\).

**Case (4)** \(g_1, g_2, g_4\) are on the top plate and \(g_3\) is on the bottom plate.

This case is equivalent to the next case.

**Case (5)** \(g_1, g_2, g_3\) are on the top plate and \(g_4\) is on the bottom plate.

Choose a red vertex \(\hat{r}\) in \(Q_n^{top} - \{r, r_1, g_4v\}\). Either \(g_2v \neq r_2\) or \(g_3v \neq r_2\). If \(g_2v \neq r_2\) then use \([2,0,2,0] = 4\) to find a 2-path covering \((g_1, \gamma_1; r_1), (g_3, \gamma_2; \hat{r})\) of \(Q_n^{top} - \{r, g_2\}\) and use \([0,0,0,2] = 4\) to find a 2-path covering \((\hat{r}v, 1; g_1), (g_2v, \gamma_3; 4)\) of \(Q_n^{bot}\). Then \((g_1, \gamma_1; r_1), (g_2, v\gamma_4; r_2), (g_3, \gamma_2v\gamma_3; g_4)\) is the required 3-path covering of \(Q_n - \{r\}\). If \(g_3v \neq r_2\) then use \([2,0,2,0] = 4\) to find a 2-path covering \((g_1, \gamma_1; r_1), (g_2, \gamma_2; \hat{r})\) of \(Q_n^{top} - \{r, g_3\}\) and use \([0,0,2,0] = 2\) to find a 2-path covering \((\hat{r}v, 1; g_1), (g_2v, \gamma_3; r_2), (g_3, \gamma_4; g_4)\) of \(Q_n^{bot}\). Then \((g_1, \gamma_1; r_1), (g_2, \gamma_2v\gamma_3; r_2), (g_3, \gamma_4; g_4)\) is the required 3-path covering of \(Q_n - \{r\}\).

**Case (6)** \(g_1, g_2\) are on the top plate and \(g_3, g_4\) are on the bottom plate.

Choose a green vertex \(g\) in \(Q_n^{top} - \{g_1, g_2\}\) such that \(gv \neq r_2\). Use \([1,1,1,1] = 4\) to find a 2-path covering \((g_1, \gamma_1; r_1), (g_2, \gamma_2; g)\) of \(Q_n^{top} - \{r\}\). Use \([0,0,0,2] = 4\) to find a 2-path covering \((gv, \gamma_3; r_2), (g_3, \gamma_4; g_4)\) of \(Q_n^{bot}\). Then \((g_1, \gamma_1; r_1), (g_2, \gamma_2v\gamma_3; r_2), (g_3, \gamma_4; g_4)\) is the required 3-path covering of \(Q_n - \{r\}\).

**Case (7)** \(g_1, g_3\) are on the top plate and \(g_2, g_4\) are on the bottom plate.

This case is equivalent to the next case.

**Case (8)** \(g_1, g_4\) are on the top plate and \(g_2, g_3\) are on the bottom plate.

Choose a green vertex \(g\) in \(Q_n^{top} - \{g_1, g_4\}\) such that \(gv \neq r_2\). Use \([1,1,1,1] = 4\) to find a 2-path covering \((g_1, \gamma_1; r_1), (g_2, \gamma_2; g_4)\) of \(Q_n^{top} - \{r\}\). Use \([0,0,2,0] = 2\) to find a 2-path covering \((g_3, \gamma_3; gv), (g_2, \gamma_4; r_2)\) of \(Q_n^{bot}\). Then \((g_1, \gamma_1; r_1), (g_2, \gamma_4; r_2), (g_3, \gamma_3v\gamma_2; g_4)\) is the required 3-path covering of \(Q_n - \{r\}\).

**Case (9)** \(g_2, g_3\) are on the top plate and \(g_1, g_4\) are on the bottom plate.

This case is equivalent to the next case.

**Case (10)** \(g_2, g_4\) are on the top plate and \(g_1, g_3\) are on the bottom plate.

Either \(g_2v \neq r_2\) or \(g_4v \neq r_2\).

If \(g_2v \neq r_2\) then use \([2,0,1,0] = 4\) to find a Hamiltonian path \((r_1, \gamma; g_4)\) of \(Q_n^{top} - \{r, g_2\}\). Since \(n \geq 5\), the length of this path is at least 13,
and so there are at least 7 edges $(\hat{r}, \hat{g})$ in $\gamma$, where $\hat{r}$ is red, $\hat{g}$ is green and $\hat{r}$ is closer to $r_1$ along $\gamma$ than $\hat{g}$ is. Choose one such edge $(\hat{r}, \hat{g})$ with the property that $\hat{r}v \notin \{g_1, g_3\}$ and $\hat{g}v \neq r_2$. Let $x$ be such that $\hat{g} = \hat{r}x$ and write $\gamma = \eta x, \theta$, with $\hat{r} = r_1\eta, g_4 = \hat{g}\theta$. Use $[0, 0, 1, 2] = 4$ to find a 3-path covering $(g_1, \gamma_1; \hat{rv}), (g_2v, \gamma_2; r_2), (g_3, \gamma_3; \hat{g}v)$ of $Q_n^{\text{bot}}$. Then $(g_1, \gamma_1v\eta^R; r_1), (g_2, v\gamma_2; r_2), (g_3, \gamma_3v\theta; g_4)$ is the required 3-path covering of $Q_n - \{r\}$.

If $g_4v \neq r_2$ then use $[2, 0, 1, 0] = 4$ to find a Hamiltonian path $(r_1, \gamma; g_2)$ of $Q_n^{\text{top}} - \{r, g_4\}$. Since $n \geq 5$, the length of this path is at least 13, and so there are at least 7 edges $(\hat{r}, \hat{g})$ in $\gamma$, where $\hat{r}$ is red, $\hat{g}$ is green and $\hat{r}$ is closer to $r_1$ along $\gamma$ than $\hat{g}$ is. Choose one such edge $(\hat{r}, \hat{g})$ with the property that $\hat{r}v \notin \{g_1, g_3\}$ and $\hat{g}v \neq r_2$. Let $x$ be such that $\hat{g} = \hat{r}x$ and write $\gamma = \eta x, \theta$, with $\hat{r} = r_1\eta, g_2 = \hat{g}\theta$. Use $[0, 0, 1, 2] = 4$ to find a 3-path covering $(g_1, \gamma_1; \hat{rv}), (\hat{g}v, \gamma_2; r_2), (g_3, \gamma_3; g_4v)$ of $Q_n^{\text{bot}}$. Then $(g_1, \gamma_1v\eta^R; r_1), (g_2, \theta^Rv\gamma_2; r_2), (g_3, \gamma_3v; g_4)$ is the required 3-path covering of $Q_n - \{r\}$.

Case (11) $g_3, g_4$ are on the top plate and $g_1, g_2$ are on the bottom plate.

Choose a green vertex $g$ in $Q_n^{\text{top}} - \{g_3, g_4\}$ such that $gv \neq r_2$. Use $[1, 1, 1, 1] = 4$ to find a 2-path covering $(g, \gamma_1; r_1), (g_3, \gamma_2; g_4)$ of $Q_n^{\text{top}} - \{r\}$. Use $[0, 0, 2, 0] = 2$ to find a 2-path covering $(g_1, \gamma_3; gv), (g_2, \gamma_4; r_2)$ of $Q_n^{\text{bot}}$. Then $(g_1, \gamma_3v\gamma_1; r_1), (g_2, \gamma_4; r_2), (g_3, \gamma_5; g_4)$ is the required 3-path covering of $Q_n - \{r\}$.

Case (12) $g_1$ is on the top plate and $g_2, g_3, g_4$ are on the bottom plate.

Choose two distinct green vertices $g$ and $\hat{g}$ in $Q_n^{\text{top}} - \{g_1\}$ such that $gv \neq r_2$ and $\hat{g}v \neq r_2$. Use $[1, 1, 1, 1] = 4$ to find a 2-path covering $(g_1, \gamma_1; r_1), (g, \gamma_2; \hat{g})$ of $Q_n^{\text{top}} - \{r\}$. Use $[0, 0, 1, 2] = 4$ to find a 3-path covering $(g_2, \gamma_3; gv), (\hat{g}v, \gamma_4; r_2), (g_3, \gamma_5; g_4)$ of $Q_n^{\text{bot}}$. Then $(g_1, \gamma_1; r_1), (g_2, \gamma_2v\gamma_4; r_2), (g_3, \gamma_5; g_4)$ is the required 3-path covering of $Q_n - \{r\}$.

Case (13) $g_2$ is on the top plate and $g_1, g_3, g_4$ are on the bottom plate.

Choose two distinct green vertices $g$ and $\hat{g}$ in $Q_n^{\text{top}} - \{g_1\}$ such that $gv \neq r_2$ and $\hat{g}v \neq r_2$. Use $[1, 1, 1, 1] = 4$ to find a 2-path covering $(g, \gamma_1; r_1), (g_2, \gamma_2; \hat{g})$ of $Q_n^{\text{top}} - \{r\}$. Use $[0, 0, 1, 2] = 4$ to find a 3-path covering $(g_1, \gamma_3; gv), (\hat{g}v, \gamma_4; r_2), (g_3, \gamma_5; g_4)$ of $Q_n^{\text{bot}}$. Then $(g_1, \gamma_3v\gamma_1; r_1), (g_2, \gamma_2v\gamma_4; r_2), (g_3, \gamma_5; g_4)$ is the required 3-path covering of $Q_n - \{r\}$.

Case (14) $g_3$ is on the top plate and $g_1, g_2, g_4$ are on the bottom plate.

This case is equivalent to the next case.

Case (15) $g_4$ is on the top plate and $g_1, g_2, g_3$ are on the bottom plate.
Let $x \neq v$ be a letter such that $g_3 xv \neq g_4$ and $\hat{r} \neq r_2$ be a red vertex in $Q_n^{\text{bot}}$ such that $\hat{r}v \neq g_4$. Use $[1, 1, 1, 1] = 4$ to find a 2-path covering $(\hat{r}v, \gamma_1; r_1)$, $(g_3 xv, \gamma_2; g_4)$ of $Q_n^{\text{top}} - \{r\}$. Use $[2, 0, 2, 0] = 4$ to find a 2-path covering $(g_1, \gamma_3, \hat{r})$, $(g_2, \gamma_4; r_2)$ of $Q_n^{\text{bot}} - \{g_3, g_3 x\}$. Then $(g_1, \gamma_3 \gamma_1; r_1), (g_2, \gamma_4; r_2), (g_3, xv \gamma_2; g_4)$ is the required 3-path covering of $Q_n - \{r\}$.

**Case (16)** $g_1, g_2, g_3, g_4$ are on the bottom plate.

Use $[1, 1, 0, 1] = 2$ to find a Hamiltonian path $(g_1, \gamma; g_2)$ of $Q_n^{\text{bot}} - \{r_2\}$. By renumbering $g_3$ and $g_4$, if necessary, we can assume that $\gamma = \eta \theta \zeta$ with $g_3 = g_1 \eta$, $g_4 = g_3 \theta$. Use $[1, 1, 1, 1] = 4$ to find a 2-path covering $(g_1 \eta v, \gamma_1; r_1)$, $(g_2 (\zeta R) v, \gamma_2; r_2 v)$ of $Q_n^{\text{top}} - \{r\}$. Then $(g_1, \eta v \gamma_1; r_1), (g_2, (\zeta R) v \gamma_2 v; r_2), (g_3, \theta; g_4)$ is the required 3-path covering of $Q_n - \{r\}$.

**References**
