ON SEQUENTIAL PROPERTIES OF NOETHERIAN TOPOLOGICAL SPACES

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ABSTRACT. Sequential properties of Noetherian topological spaces are considered. A topological space X is called *Noetherian* if for every increasing by inclusion sequence $(U_n)_{n=1}^{\infty}$ of open subsets of X there exists n such that $U_n = U_{n+1} = \ldots$. It is shown that every Noetherian topological space is sequentially compact and that the sequential topology inherits the Noetherian property. Hence, every sequentially open cover of a Noetherian topological space has a finite subcover. The following result is proved: Let X be a Noetherian topological space in which every irreducible closed subset F has a generic point. The space X is sequential if and only if $h(X) \leq \omega_1$, where $h(X)$ is a suitable ordinal invariant. From this result follows that a Zariski space X is sequential if and only if $h(X) \leq \omega_1$ and that if R is a commutative Noetherian ring then the prime spectrum $\operatorname{Spec} R$ is a sequential Noetherian topological space.

1. INTRODUCTION

The concept of Noetherian topological spaces arises naturally in the study of Noetherian rings and is of great interest in some areas of mathematics such as Algebraic Geometry. A topological space (X, τ) is called *Noetherian* if τ satisfies the *ascending chain condition* (a.c.c. for short): every strictly ascending chain $U_1 \subset U_2 \subset \cdots$ of elements of τ is finite (see [1], [2], [7], [9]).

Topological spaces that satisfy properties similar to a.c.c. have been widely studied. In [8], spaces with Noetherian bases have been introduced (a topological space has a *Noetherian base* if it has a base that satisfies a.c.c.) and many interesting results about such spaces have been obtained (see [8], [6], [14]). Clearly, every Noetherian space has a Noetherian base but the converse is not true in general. For example, every Hausdorff Noetherian topological space is finite, whereas it is still unknow if there exists in ZFC a Hausdorff space without a Noetherian base.

We have to mention here that some topologists (see [10], [12]) define a space to be Noetherian if it has a base $\mathfrak{B}$ such that for every $G \in \mathfrak{B}$ the

cardinality of the set $\{B \in \mathfrak{B} : G \subset B\}$ is finite. Obviously, every such base is Noetherian and therefore the class of spaces defined in that way is different from the Noetherian spaces considered here.

In this paper we study sequential properties of Noetherian topological spaces. It is shown that every Noetherian topological space is sequentially compact and s-compact. The concept of s-compactness was introduced and studied by the first author in [5]: A topological space X is *s-compact* if every sequentially open cover of X has a finite subcover. In the same paper [5] the following characterization of the sequential compactness is given: A T_0-topological space X is sequentially compact if and only if every countable sequentially open cover of X has a finite subcover. For the sake of completeness and easy reference, this theorem and its proof are given in Section 3 (Theorem 3.6).

In order to avoid possible confusion, we have to point out that R. Prasad and R. S. Yadav [13] (and some other authors) have used the term *s-compact* for spaces that satisfy the following property: every cover of semi-open subsets of X has a finite subcover (a subset A of a topological space X is called *semi-open* if there is an open set $U \subset X$ such that $U \subset A \subset \overline{U}$). The same class of spaces has also been considered independently by C. Dorsett in [3] under the name of *semicompact spaces*. Dorsett's paper appeared about one year earlier than [13] (see MR0724336 (85d:54022)) and nowadays most of the topologists use the name *semicompact spaces* for such spaces. It is worth while to mention that despite of the fact that the class of s-compact spaces (in our sense of the word) and the class of semicompact spaces contain only compact spaces, both classes are different. The space in Example 3.10 is a semicompact space which is not s-compact and the space defined in Example 3.2 in [13] is an s-compact space which is not semicompact.

The main result in this paper, contained in Theorem 4.7, is the following: Let X be a Noetherian topological space in which every irreducible closed subset F has a generic point. The space X is sequential if and only if $\mathrm{h}(X) \leq \omega_1$ (for the definition of $\mathrm{h}(X)$ see Definition 2.12). From this result follows that a Zariski space X is sequential if and only if $\mathrm{h}(X) \leq \omega_1$ and that if R is a commutative Noetherian ring then the prime spectrum $\operatorname{Spec} R$ is a sequential Noetherian topological space.

The preliminary definitions and results about Noetherian spaces and irreducible spaces are given in Section 2 and those of sequential type – in Section 3. All results about sequential properties of Noetherian topological spaces are contained in the last section. For terminology and notation not given here see [4], [15], or [2]. Throughout the paper, topological space means T_0-space, $\mathbb{Z}$ denotes the set of all integers, and $\mathbb{N}$ – the set of all positive integers.

2. Noetherian and Irreducable Spaces

The notion of Noetherian rings is fundamental in Algebraic Geometry and Noetherian rings have been extensively studied (see [1], [2], [7], [9]).

Definition 2.1. [1] A commutative ring R is called *Noetherian* if every increasing by inclusion sequence of ideals of R is stationary.

With every commutative ring R is associated a topological space $\operatorname{Spec} R$, the prime spectrum of R.

Definition 2.2. [1] Let R be a commutative ring. $\operatorname{Spec} R$ is the set of all prime ideals of R provided with the Zariski topology in which $F \subset \operatorname{Spec} R$ is closed if and only if there exists an ideal I of R such that $F = \{p \in \operatorname{Spec} R \mid p \supseteq I\}$.

The space $\operatorname{Spec} R$ is a T_0-topological space [9, Proposition 1.5]. Also, if R is a Noetherian ring then the topology on $\operatorname{Spec} R$ satisfies the ascending chain condition. Therefore, the following definition is natural.

Definition 2.3. [2] A topological space (X, τ) is called *Noetherian* if τ satisfies a.c.c., or equivalently, for every decreasing by inclusion sequence $(F_n)_{n=1}^{\infty}$ of closed subsets of X there exists n such that $F_n = F_{n+1} = \ldots$.

Now, we can restate the above-mentioned property of $\operatorname{Spec} R$ as follows.

Theorem 2.4. [2, II.4.3, Corollary 7]] *If R is a Noetherian ring then $\operatorname{Spec} R$ is a Noetherian topological space.*

Definition 2.5. A topological space X is called *compact* if every open cover of X has a finite subcover.

The following theorem contains some of the basic properties of the Noetherian topological spaces.

Theorem 2.6. [2, II.4.2], [7, Exercise 1.7]
  (a) *Every Noetherian topological space is compact.*
  (b) *Every subspace of a Noetherian topological space is Noetherian.*
  (c) *A topological space X is Noetherian if and only if every open subset of X is compact.*
  (d) *If X is a topological space and $X = F_1 \cup F_2 \cup ... \cup F_n$, where each F_i is a Noetherian subspace of X, then X is Noetherian.*
  (e) *A Noetherian topological space is Hausdorff if and only if it is a finite set with the discrete topology.*
  (f) *A continuous image of a Noetherian space is Noetherian.*

Definition 2.7. [2] A topological space X is called *irreducible* if the intersection of any finite collection of non-empty open subsets of X is not empty, or equivalently, if X cannot be written as a finite union of closed proper subsets. A subset Y of X is called *irreducible* if Y is irreducible as a topological space with the relative topology. A maximal irreducible subset of X

is called an *irreducible component* of X. The empty set is not considered to be irreducible.

Some fundamental properties of the irreducible topological spaces are stated in the following theorem.

Theorem 2.8. [2, II.4.1], [9, 1.6] *Let X be a topological space and Y be a subset of X.*

(a) *X is irreducible if and only if every non-empty open subset of X is dense in X.*

(b) *X is irreducible if and only if every open subset of X is connected.*

(c) *If X is an irreducible space then every non-empty open subset of X is irreducible.*

(d) *The closure $\overline{\{x\}}$ of a point x is irreducible.*

(e) *Every irreducible subset of X is contained in an irreducible component of X and X is the union of its irreducible components.*

(f) *Y is irreducible if and only if its closure $\overline{Y}$ is irreducible.*

(g) *Every irreducible component of Y is closed in Y.*

(h) *If Y has only a finite number of distinct irreducible components F_i, $i = 1, 2, \ldots, n$ then the irreducible components of the closure $\overline{Y}$ in X are the closures $\overline{F_i}$ of the F_i, $i = 1, 2, \ldots, n$ and $\overline{F_i} \neq \overline{F_j}$ for $i \neq j$.*

(i) *A Hausdorff space X is irreducible if and only if it consists of a single point.*

(j) *A continuous image of an irreducible space is irreducible.*

The following theorem sheds light on the relationship between Noetherian topological spaces and irreducible spaces.

Theorem 2.9. [7, Proposition 1.5] *If X is a Noetherian topological space then there exists a natural number n and unique irreducible components F_1, $F_2, \ldots, F_n$ of X such that $X = F_1 \cup F_2 \cup \ldots \cup F_n$ and $F_i \nsubseteq F_j$ for all $i \neq j$.*

Clearly, any Noetherian space with more than one irreducible component is not an irreducible space. All finite discrete spaces with more than one point are such spaces. Also, a topological space could be irreducible without being Noetherian. Every uncountable set with the co-countable topology is an example of such a space.

Definition 2.10. Let X be a topological space. If $X = \overline{\{x\}}$ for some $x \in X$, then x is said to be a *generic point* of X.

Theorem 2.11. [7, Exercise 2.9] *Let R be a commutative ring. Then every irreducible closed subset $F \subset \operatorname{Spec} R$ has a unique generic point.*

Next, we introduce the notion of *hight* of a Noetherian topological space.

Definition 2.12. Let X be a non-empty Noetherian topological space and $\mathfrak{P}(X)$ be the set of all irreducible closed subsets of X, ordered by inclusion. Let α be the supremum of all ordinals β such that there exists a strictly

increasing function $P : [0, \beta) \rightarrow \mathfrak{P}(X)$. We shall say that the *hight* of X is α (denoted $h(X) = \alpha$) if α is an infinite ordinal and $h(X) = \alpha - 1$ otherwise. We define the hight of the empty set to be -1.

It is clear that if $h(X) < \omega_0$ then $h(X) = \dim X$, where $\dim X$ is the *dimension* of the Noetherian topological space X defined to be the supremum of all integers n such that there exists a chain $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ of distinct irreducible closed subsets of X (see [7, I.1]). The notion of dimension of a Noetherian topological space has the following natural generalization.

Definition 2.13. Let X be a Noetherian topological space and $\mathfrak{P}(X)$ be the set of all irreducible closed subsets of X and the empty set. $\dim X$ is defined inductively as follows.

    (1) $\dim X = -1$ if and only if $X = \emptyset$.
    (2) If α is an ordinal number then $\dim X \leq \alpha$ if for every $F \in \mathfrak{P}(X)$ and for every $G \subset F$ such that $G \neq F$ and $G \in \mathfrak{P}(X)$ we have $\dim G < \alpha$.
    (3) $\dim X = \min\{\alpha | \dim X \leq \alpha\}$.

In the same way one could define $\dim X$ for any topological space. However, for non-Noetherian spaces $\dim X$ is not always useful. For example, $\dim X = 0$ for any Hausdorff space X. Also, it can be verified that for Noetherian topological spaces $\dim X = \operatorname{ind} X$, where $\operatorname{ind} X$ is the *small inductive dimension* of X (see [11]). Therefore, if X is a Noetherian topological space and $h(X) < \omega_0$ then $h(X) = \dim X = \operatorname{ind} X$. Of course, $h(X)$ and $\dim X$ are not always equal. For the space X, from Example 4.5, we have $h(X) = \omega_1 + 1$ and $\dim X = \omega_1$. Hence, there exists a Noetherian topological space X for which $h(X) > \dim X$. On the other hand, for the space Y, defined in the following example, we have $h(Y) = \omega_0$ but $\dim Y = \omega_0 + 1$. Thus, there exists a Noetherian topological space for which $h(Y) < \dim Y$.

Example 2.14. *For every* $n \in \mathbb{N}$ *let* $X_n = \{i | 0 \leq i \leq n, i \in \mathbb{Z}\}$ *provided with the topology where the non-empty closed sets are all intervals* $[0, m]$, $0 \leq m \leq n$. *Let* X *be the disjoint union of all* X_n, $n \in \mathbb{N}$ *with the topology in which closed sets are* $\emptyset$, X, *and all unions* $\bigcup_{i=1}^{k} Y_i$, *where* Y_i *is a closed subset of* X_i *for every* $i = 1, \ldots, k$, $k \in \mathbb{N}$. *With this topology* X *is an irreducible Noetherian topological space for which* $h(X) = \dim X = \omega_0$. *Now, let* $Y = X \cup \{\infty\}$ *provided with the topology in which closed sets are all closed sets in* X *and the set* Y. *With this topology* Y *is an irreducible Noetherian topological space for which* $\dim Y = \omega_0 + 1$ *but* $h(Y) = \omega_0$.

At the end of this section we provide one more fact about Noetherian rings. Its corollary shall be used in the last section.

Theorem 2.15. [1, Corollary 11.12] *In a Noetherian ring every decreasing sequence of prime ideals is stationary.*

Corollary 2.16. *If* R *is a Noetherian ring then* $h(\operatorname{Spec} R) \leq \omega_0$.

Proof. It follows from Theorem 2.15, Definition 2.2, and Definition 2.12. $\square$

3. Properties of Sequential Type

We begin this section with the definition of sequentially closed and sequentially open sets.

Definition 3.1. [15] A subset A of a topological space X is called *sequentially closed* if it has the following property: if a sequence in A converges in X to a point x, then $x \in A$. A subset E of a topological space X is called *sequentially open* if $X \backslash E$ is sequentially closed.

For every $A \subset X$, we denote by $\overline{A}^s$ the sequential closure of A in X, which is the minimal sequentially closed set in X that contains A, and by $[A]_s$ – the set A together with all limits of convergent sequences of points from the set A.

Definition 3.2. [5] A cover of a topological space X is called *sequentially open* if its elements are sequentially open sets.

Definition 3.3. [15] Let (X, τ) be a topological space. The *sequential topology* τ_s is the topology on X such that a subset E of X is open in (X, τ_s) if and only if E is sequentially open in (X, τ).

Definition 3.4. [15] A topological space X is called *countably compact* if every countable open cover of X has a finite subcover.

Definition 3.5. [15] A topological space X is called *sequentially compact* if every sequence of points of X has a convergent subsequence.

The following theorem characterizes sequential compactness in terms of sequentially open covers.

Theorem 3.6. [5] *For a T_0-topological space X the following conditions are equivalent:*

(a) *X is a sequentially compact space.*
(b) *Every countable sequentially open cover of X has a finite subcover.*
(c) *(X, τ_s) is a countably compact space.*

Proof. First we shall prove that (a) implies (b). Let us assume that there exists a sequentially compact space X which has a countable sequentially open cover $\{U_i\}_{i=1}^{\infty}$ without finite subcovers. Then we can find a sequence of distinct points $(x_n)_{n=1}^{\infty}$ such that for every $n \in \mathbb{N}$, $x_n \notin \bigcup_{i=1}^{n} U_i$. Let $(x_{n_k})_{k=1}^{\infty}$ be a convergent subsequence of $(x_n)_{n=1}^{\infty}$ and x be one of its limit points. Since $\{U_i\}_{i=1}^{\infty}$ is a cover of X, there exists $m \in \mathbb{N}$ such that $x \in U_m$. The set U_m contains no more than finitely many elements of the sequence $(x_n)_{n=1}^{\infty}$. Thus, almost all but finitely many elements of the sequence $(x_{n_k})_{k=1}^{\infty}$ belong to $X \backslash U_m$, which is a sequentially closed set. Therefore $x \in X \backslash U_m$ – a contradiction.

We shall now prove that (b) implies (a). Let us assume that there exists a T_0-space X which is not sequentially compact but every sequentially open cover of X has a finite subcover. Then there exists a sequence $(x_n)_{n=1}^{\infty}$ in X

without convergent subsequences. If x is an arbitrary point in X then there are no more than finitely many $n \in \mathbb{N}$ such that $x \in \overline{\{x_n\}}$. For, if there were infinitely many such x_n then they would form a subsequence of $(x_n)_{n=1}^{\infty}$ that would approach to x. Therefore, we can choose a subsequence $(x_{n_k})_{k=1}^{\infty}$ of distinct points of $(x_n)_{n=1}^{\infty}$ such that $\{x_{n_1}, x_{n_2}, \ldots, x_{n_k}\} \cap \overline{\{x_{n_{k+1}}\}} = \emptyset$ for every $k \in \mathbb{N}$.

Now, for every $m \in \mathbb{N}$, let $A_m = \{x_{n_k}\}_{k=m+1}^{\infty}$, $B_m = [A_m]_s$, and $C_m = [B_m]_s$. We shall show that $B_m = C_m$. We need only to prove that $C_m \subset B_m$. Let $x \in C_m$. Then we can find a sequence $(y_i)_{i=1}^{\infty}$ of points from B_m that approaches x. However, $(x_{n_k})_{k=1}^{\infty}$ does not have convergent subsequences. Thus for every y_i we can find $z_i \in A_m$ such that $y_i \in \overline{\{z_i\}}$. It is easily seen that the sequence $(z_i)_{i=1}^{\infty}$ converges to x. If we assume that this sequence has infinitely many distinct points then we would be able to choose a subsequence of it, which would also be a convergent subsequence of $(x_{n_k})_{k=1}^{\infty}$ – a contradiction. Hence, the sequence $(z_i)_{i=1}^{\infty}$ has only finitely many distinct points. Thus, there exists $p \in \mathbb{N}$ such that $x \in \overline{\{z_p\}}$ and since $z_p \in A_m$ we have $x \in B_m$. Therefore $B_m = C_m$, which means that $B_m = \overline{\{x_{n_k}\}_{k=m+1}^{\infty}}^s$.

Next, let us note that if $U_m = X \backslash B_m$ then $x_{n_m} \in U_m$ for every $m \in \mathbb{N}$. For, if we assume that $x_{n_m} \notin U_m$ for some $m \in \mathbb{N}$, or equivalently, $x_{n_m} \in B_m$, then there would exist $k \geq m+1$, $k \in \mathbb{N}$ such that $x_{n_m} \in \overline{\{x_{n_k}\}}$, which would contradict with the choice of the sequence $(x_{n_k})_{k=1}^{\infty}$.

We shall now verify that $\{U_m\}_{m=1}^{\infty}$ is a cover of X. Indeed, if we assume that there exists $x \in X$ such that $x \notin U_m$ for every $m \in \mathbb{N}$ then $x \in B_m$ for every $m \in \mathbb{N}$. Thus, for every $m \in \mathbb{N}$, we can choose $k_m \geq m + 1$, $k_m \in \mathbb{N}$ such that $x \in \overline{\{x_{n_{k_m}}\}}$. The resulting sequence $(x_{n_{k_m}})_{m=1}^{\infty}$ is a sequence that approaches x and contains a convergent subsequence, which is a subsequence of $(x_n)_{n=1}^{\infty}$ – a contradiction. Therefore $\{U_m\}_{m=1}^{\infty}$ is a sequentially open cover of X without finite subcovers, which is a contradiction.

The fact that (b) and (c) are equivalent follows directly from the definitions. $\qquad\square$

Definition 3.7. [15] A topological space (X, τ) is called *sequential* if $\tau = \tau_s$.

Definition 3.8. [5] A topological space (X, τ) is called *s-compact* if (X, τ_s) is compact, or equivalently, every sequentially open cover of X has a finite subcover.

Theorem 3.9. [5]
   (a) *For the class of sequential spaces s-compactness coincides with compactness.*
   (b) *Every s-compact space is a compact space.*
   (c) *Every s-compact space is a sequentially compact space.*

Example 3.10. [5] *Let ω_1 be the first uncountable ordinal. The topological space $[0, \omega_1]$ with the usual ordered topology is an example of a compact and sequentially compact space, which is not s-compact.*

4. Main Results

We begin this section with the following fundamental lemma.

Lemma 4.1. *Let X be a Noetherian topological space. Every sequence $(x_n)_{n=1}^\infty$ in X has a convergent subsequence $(x_{n_k})_{k=1}^\infty$ such that:*

(a) *The set of all limit points $\lim_{k\to\infty} x_{n_k}$ is equal to the set $\overline{\{x_{n_k}\}_{k=1}^\infty}$.*

(b) *The set $\overline{\{x_{n_k}\}_{k=1}^\infty}$ is irreducible.*

Proof. Let X be a Noetherian topological space and $(x_n)_{n=1}^\infty$ be a sequence in X. For every infinite subset $I = \{m_1, m_{2,...}\mid 0 < m_1 < m_2 < ...\}$ of the set $\mathbb{N} = \{1, 2, ..., n, ...\}$ let $(x_m)_{m\in I} = (x_{m_k})_{k=1}^\infty$ and $Y_I = \overline{\{x_m\}_{m\in I}}$. Then $\{Y_I\}_{I\subset\mathbb{N}}$ is a non-empty family of closed subsets of the Noetherian topological space X and therefore it has a minimal element Y_J, $J = \{n_1, n_2, ..., n_k, ...\}$.

(a) Let $y \in Y_J$, U be an open neighborhood of y, and let us suppose that $(x_{n_k})_{k=1}^\infty$ does not converge to y. Then the set $J_U = \{j \in J \mid x_j \notin U\}$ is an infinite set and $J_U \subset J \subset \mathbb{N}$. Hence $Y_{J_U} \subset Y_J$ and because Y_J is a minimal element of the family $\{Y_I\}_{I\subset\mathbb{N}}$, we have $Y_{J_U} = Y_J$. Thus $y \in Y_{J_U}$ and since $Y_{J_U} = \overline{\{x_j \mid j \in J_U\}}$ it follows that $U \cap \{x_j \mid j \in J_U\} \neq \emptyset$ - a contradiction. Therefore, the set J_U is finite and consequently $y \in \lim_{k\to\infty} x_{n_k}$. This means that $Y_J \subset \lim_{k\to\infty} x_{n_k}$ and since $\lim_{k\to\infty} x_{n_k} \subset Y_J$ then $Y_J = \lim_{k\to\infty} x_{n_k}$.

(b) Let $m \in \mathbb{N}$ and $U_1, U_2, ..., U_m$ be non-empty open sets in $Y_J = \overline{\{x_{n_k}\}_{k=1}^\infty}$. Then there exist open sets $V_1, V_2, ..., V_m$ in X such that $U_i = Y_J \cap V_i$ for $i = 1, 2, .., m$. It follows from (a) that $J_{V_i} = \{j \in J \mid x_j \notin V_i\}$ is finite for each $i = 1, 2, ..., m$. Hence, there exists k such that $x_{n_k} \in Y_J \cap V_1 \cap V_2 \cap ... \cap V_m = U_1 \cap U_2 \cap ... \cap U_m$. Therefore, the set $\overline{\{x_{n_k}\}_{k=1}^\infty}$ is irreducible. $\square$

Theorem 4.2. *Every Noetherian topological space is sequentially compact.*

Proof. It follows immediately from Lemma 4.1. $\square$

Theorem 4.3. *Let (X, τ) be a Noetherian topological space. Then (X, τ_s) is a Noetherian topological space.*

Proof. Let us suppose that (X, τ) is a Noetherian topological space but (X, τ_s) is not Noetherian. Then there exists a strictly decreasing by inclusion sequence $F_1 \supset F_2 \supset ... \supset F_n \supset ...$ of distinct sequentially closed subsets of X. For each $n \in \mathbb{N}$ we choose a point $x_n \in F_n \backslash F_{n+1}$ and we form the sequence $(x_n)_{n=1}^\infty$. According to Lemma 4.1, this sequence has a convergent subsequence $(x_{n_k})_{k=1}^\infty$ such that the set of all its limit points $\lim_{k\to\infty} x_{n_k}$ is equal to the set $\overline{\{x_{n_k}\}_{k=1}^\infty}$. Then $x_{n_1} \in \lim_{k\to\infty} x_{n_k}$ and therefore $x_{n_1} \in \lim_{k\to\infty} x_{n_{k+1}}$. However, the set $\{x_{n_2}, x_{n_3}, ..., x_{n_{k+1}}, ...\}$ is a subset of the sequentially closed set F_{n_2}. Thus, $\lim_{k\to\infty} x_{n_{k+1}} \subset F_{n_2}$ and hence $x_{n_1} \in F_{n_2}$. This is a contradiction because $x_{n_1} \in F_{n_1}\backslash F_{n_1+1} \subset F_{n_1}\backslash F_{n_2}$. $\square$

Corollary 4.4. *Every Noetherian topological space is s-compact.*

Proof. It follows immediately from Theorem 4.3 and Definition 3.8.   □

Since every compact sequential space is s-compact (Theorem 3.9), it is natural to ask whether every Noetherian topological space is sequential. The following two examples show that it is not true in general.

Example 4.5. *Let ω_1 be the first uncountable ordinal and X be the interval $[0, \omega_1]$ with the following topology: a non-empty set F is closed if and only if $F = [0, \alpha]$ for some $\alpha \in [0, \omega_1]$. Then X is Noetherian but it is not a sequential topological space.*

In Example 4.5, the space X is uncountable with hight $h(X) = \omega_1 + 1$. In the following example, the space Y is a countable Noetherian topological space with hight $h(Y) = 2$ which is not sequential, either.

Example 4.6. *Let $Y = \{0, 1, 2, ..., n, ...\}$ and $\mathfrak{F}$ be an infinite maximal almost disjoint family of infinite subsets of $\mathbb{N} = \{1, 2, ..., n, ...\}$. We denote by $\mathfrak{C}$ the set with the following elements: $\varnothing$, Y, all finite sets $D \subset \mathbb{N}$, and all sets of the form $D \cup \bigcup_{i=1}^{n} F_i$, where $n \in \mathbb{N}$ and $F_i \in \mathfrak{F}$ for $i = 1, 2, ..., n$. Y is provided with the topology in which $\mathfrak{C}$ is the family of all closed sets. Then $h(Y) = 2$ and Y is Noetherian but it is not a sequential topological space.*

It follows from the above examples that a given Noetherian topological space X could be non-sequential regardless of its hight. The following theorem shows that for a very large class of Noetherian topological spaces the sequentiality of X depends only on the value of $h(X)$.

Theorem 4.7. *Let X be a Noetherian topological space in which every irreducible closed subset F has a generic point. The space X is sequential if and only if $h(X) \leq \omega_1$.*

Proof. First, we shall prove that if $h(X) \leq \omega_1$ then X is a sequential space. It is sufficient to prove that if $Z \subset X$ is a sequentially closed set then $Z = \overline{Z}$. According to Theorem 2.6, Z is a Noetherian space and therefore has a finite number of irreducible components (Theorem 2.9). Let $Z = F_1 \cup F_2 \cup ... \cup F_n$, where F_i, $i = 1, 2, ..., n$ are the irreducible components of Z. Then $\overline{Z} = \overline{F_1} \cup \overline{F_2} \cup ... \cup \overline{F_n}$ and according to Theorem 2.8, $\overline{F_1}, \overline{F_2}, ..., \overline{F_n}$ are the irreducible components of $\overline{Z}$. Thus, to prove that $Z = \overline{Z}$, it is sufficient to prove that $F_i = \overline{F_i}$ for every $i = 1, 2, ..., n$. However, for every $i = 1, 2, ..., n$, F_i is a sequentially closed set in X and $\overline{F_i}$ is an irreducible subset of X. Therefore, to complete the proof, it is sufficient to prove that if Z is a sequentially closed set such that $\overline{Z}$ is an irreducible subset of X, then $Z = \overline{Z}$.

Now, we suppose that Z is such a set and $Z \neq \overline{Z}$. Let $\mathfrak{P}(Z)$ be the partially ordered by inclusion set of all closed in X irreducible subsets of Z. We shall prove that every point $z \in Z$ belongs to a maximal by inclusion element of $\mathfrak{P}(Z)$.

We consider an arbitrary chain $\mathfrak{C}$ of elements of $\mathfrak{P}(Z)$. $\mathfrak{C}$ is well ordered because X is a Noetherian topological space. Then there exists ordinal α and

strictly increasing bijection $P : [0, \alpha) \to \mathfrak{C}$. Clearly, $\alpha \leq \omega_1$ because $\mathrm{h}(X) \leq \omega_1$. We can extend P setting $P(\alpha) = \overline{Z}$. Thus, if we assume that $\alpha = \omega_1$, we get $\mathrm{h}(\overline{Z}) \geq \omega_1 + 1$, which is a contradiction. Consequently $\alpha < \omega_1$. Hence, $\mathfrak{C}$ is a countable set. Therefore, there exists a sequence $C_1, C_2, ..., C_n, ...$ of elements of $\mathfrak{C}$ such that $C_n \subseteq C_{n+1}$ and for every $C \in \mathfrak{C}$ there exists a number n such that $C \subseteq C_n$. For every $n \in \mathbb{N}$ we choose z_n to be the generic point for C_n. It follows from Lemma 4.1 that the sequence $(z_n)_{n=1}^\infty$ has a convergent subsequence $(z_{n_k})_{k=1}^\infty$ such that $\lim_{k \to \infty} z_{n_k} = \overline{\{z_{n_k}\}_{k=1}^\infty}$ and the set $\overline{\{z_{n_k}\}_{k=1}^\infty}$ is irreducible. If $C \in \mathfrak{C}$ then there exists an integer n such that $C \subseteq C_n$. Let us choose a number $m \in \mathbb{N}$ such that $n_m \geq n$. Then $C \subseteq C_n \subseteq C_{n_m} = \overline{\{z_{n_m}\}} \subseteq \overline{\{z_{n_k}\}_{k=1}^\infty} \subset Z$. This means that every chain $\mathfrak{C}$ in $\mathfrak{P}(Z)$ has an upper bound. Now, applying Zorn's lemma, we conclude that every element of $\mathfrak{P}(Z)$ can be included in a maximal one. Obviously, if $z \in Z$ then $\overline{\{z\}} \subset Z$, hence $\overline{\{z\}} \in \mathfrak{P}(Z)$. Therefore, every point $z \in Z$ belongs to a maximal element of $\mathfrak{P}(Z)$.

Let us assume that $\mathfrak{P}(Z)$ has infinitely many maximal elements. Therefore, we can choose a sequence $B_1, B_2, ..., B_n, ...$ of distinct maximal elements of $\mathfrak{P}(Z)$. For every $n \in \mathbb{N}$, we choose z_n to be the generic point of B_n. According to Lemma 4.1, the sequence $(z_n)_{n=1}^\infty$ has a convergent subsequence $(z_{n_k})_{k=1}^\infty$ such that $\lim_{k \to \infty} z_{n_k} = \overline{\{z_{n_k}\}_{k=1}^\infty}$ and the set $\overline{\{z_{n_k}\}_{k=1}^\infty}$ is irreducible, hence $\overline{\{z_{n_k}\}_{k=1}^\infty} \in \mathfrak{P}(Z)$. Therefore, there exists a maximal, irreducible, closed in X, set B such that $B \in \mathfrak{P}(Z)$ and $\overline{\{z_{n_k}\}_{k=1}^\infty} \subset B$. Also, for every $m \in \mathbb{N}$ we have $B_{n_m} = \overline{\{z_{n_m}\}} \subseteq \overline{\{z_{n_k}\}_{k=1}^\infty} \subset B$. Now, it follows from the maximality of the sets B_{n_k} that $B = B_{n_1} = B_{n_2} = ...$, which contradicts with the choice of the sets B_n. Therefore, the maximal elements of $\mathfrak{P}(Z)$ are finitely many. Consequently, Z is a closed set in X, being a union of all maximal elements of $\mathfrak{P}(Z)$. That contradicts with $Z \neq \overline{Z}$. This completes the proof that if $\mathrm{h}(X) \leq \omega_1$ then X is a sequential space.

Now, we shall prove that if $\mathrm{h}(X) > \omega_1$ then X is not a sequential space. Let $\mathfrak{P}(X)$ be the partially ordered by inclusion set of all closed irreducible subsets of X. From the inequality $\mathrm{h}(X) > \omega_1$ readily follows that there exists a strictly increasing map $P : [0, \omega_1 + 1) \to \mathfrak{P}(X)$. We denote by Z the set $\bigcup_{\alpha < \omega_1} P(\alpha)$. We shall prove that Z is a sequentially closed subset of X which is not closed. Let $(z_n)_{n=1}^\infty$ be a convergent sequence in X with $\{z_n\}_{n=1}^\infty \subset Z$. There exists an ordinal number $\alpha < \omega_1$ such that $z_n \in P(\alpha)$ for every $n \in \mathbb{N}$. Clearly, if $z \in \lim_{n \to \infty} z_n$ then $z \in P(\alpha) \subset Z$. Therefore, Z is sequentially closed. Let us suppose that Z is closed in X and $Z = F_1 \cup F_2 \cup ... \cup F_m$, where $m \in \mathbb{N}$ and the closed in X sets $F_1, F_2, ..., F_m$ are the irreducible components of Z. For every F_i, $i = 1, 2, ..., m$, we choose its generic point $z_i \in F_i$. Let $z_i \in P(\alpha_i)$, $i = 1, 2, ..., m$ and $\alpha = \max(\alpha_1, ..., \alpha_m)$. Then $z_i \in P(\alpha)$ and therefore $F_i \subset P(\alpha)$ for every $i = 1, 2, ..., m$. Hence, $P(\alpha) = Z$ and $P(\alpha) = P(\alpha + 1) = ...$ - a contradiction. Therefore, Z is a sequentially closed subset of X which is not closed. Thus, X is not a sequential space. $\qquad \square$

Definition 4.8. [7] A topological space X is a *Zariski* space if it is Noetherian and every irreducible closed subset of X has a unique generic point.

Corollary 4.9. *A Zariski space X is sequential if and only if $\mathrm{h}(X) \leq \omega_1$.*

Proof. It follows immediately from Theorem 4.7. $\qquad\square$

Corollary 4.10. *Let R be a commutative Noetherian ring. Then the prime spectrum $\mathrm{Spec}\,R$ is a sequential Noetherian topological space.*

Proof. It follows from Theorem 2.4, Theorem 2.11, Corollary 2.16, and Theorem 4.7. $\qquad\square$

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