CARDINALITIES OF WEAKLY LINDELÖF SPACES WITH REGULAR $G_\kappa$-DIAGONALS

IVAN S. GOTCHEV

Abstract. For a Urysohn space $X$ we define the regular diagonal degree $\Delta(X)$ of $X$ to be the minimal infinite cardinal $\kappa$ such that $X$ has a regular $G_\kappa$-diagonal i.e. there is a family $(U_\eta : \eta < \kappa)$ of open neighborhoods of $\Delta_X = \{(x,x) : x \in X\}$ in $X^2$ such that $\Delta_X = \bigcap_{\eta < \kappa} U_\eta$.

In this paper we show that if $X$ is a Urysohn space then: (1) $|X| \leq 2^{\chi(X)} \cdot \Delta(X)$; (2) $|X| \leq 2^{\chi(X)} \cdot 2^{wL(X)}$; (3) $|X| \leq wL(X) \cdot \chi(X)$; and (4) $|X| \leq aL(X) \cdot \chi(X)$, where $\chi(X)$, $c(X)$, $wL(X)$ and $aL(X)$ are respectively the character, the cellularity, the weak Lindelöf number and the almost Lindelöf number of $X$.

The first inequality extends to the uncountable case Buzyakova’s result that the cardinality of a ccc-space with a regular $G_\delta$-diagonal does not exceed $2^{\omega}$. It follows from (2) that every weakly Lindelöf space with a regular $G_\delta$-diagonal has cardinality at most $2^{2^{\omega}}$.

Inequality (3) implies that when $X$ is a space with a regular $G_\delta$-diagonal then $|X| \leq wL(X) \cdot \chi(X)$. This improves significantly Bell, Ginsburg and Woods inequality $|X| \leq 2^{\chi(X) \cdot wL(X)}$ for the class of normal spaces with regular $G_\delta$-diagonals. In particular (3) shows that the cardinality of every first countable space with a regular $G_\delta$-diagonal does not exceed $wL(X)^\omega$.

For the class of spaces with regular $G_\delta$-diagonals (4) improves Bella and Cammaroto inequality $|X| \leq 2^{\chi(X) \cdot aL(X)}$, which is valid for all Urysohn spaces. Also, it follows from (4) that the cardinality of every space with a regular $G_\delta$-diagonal does not exceed $aL(X)^\omega$.

1. Introduction

Perhaps the two most famous results involving cardinal functions are Arhangel’skii’s and Hajnal-Juhász’ theorems asserting that if $X$ is a Hausdorff space then $|X| \leq 2^{\chi(X) \cdot L(X)}$ [1] and $|X| \leq 2^{\chi(X) \cdot c(X)}$ [19], where $\chi(X)$, $L(X)$ and $c(X)$ denote respectively the character, Lindelöf number and cellularity of $X$.
Bell, Ginsburg and Woods showed in [4] that if $X$ is a normal space then
\[ |X| \leq 2^{\chi(X)wL(X)} \]  
where $wL(X)$ is the weak Lindelöf number of $X$. Since $wL(X) \leq L(X)$ and $wL(X) \leq c(X)$, (1) generalizes (for the class of normal spaces) Arhangel’skii’s and Hajnal-Juhász’ inequalities. In the same paper the authors constructed an example (see [4, Example 2.3]) showing that for Hausdorff spaces the gap between $|X|$ and $2^{\chi(X)wL(X)}$ could be arbitrarily large and they asked (see [4, 4.1]) if (1) holds true for all regular $T_1$-spaces. To the best of our knowledge this question is still open (see [17, Question 1]). In this paper we give a partial answer to their question by showing that for every space $X$ with a regular $G_\delta$-diagonal even the stronger inequality $|X| \leq wL(X)\chi(X)$ is true.

In 1977, Ginsburg and Woods proved (see [12]) that if $X$ is a $T_1$-space then $|X| \leq 2^{e(X)\Delta(X)}$, where $e(X)$ and $\Delta(X)$ denote respectively the extent and the diagonal degree of $X$. As a corollary of that inequality the authors obtained that if $X$ is a collectionwise Hausdorff space then
\[ |X| \leq 2^{e(X)\Delta(X)}. \]  
They also noticed (see [12, Example 2.4]) that the Katětov extension $k\omega$ of the countable discrete space $\omega$ is an example of a Urysohn space (every two points have disjoint closed neighborhoods) for which $|k\omega| > 2^{e(k\omega)\Delta(k\omega)}$ and they asked if (2) was true for every regular $T_1$-space [12, Question 2.5]. In 1978 Arhangel’skii independently asked the countable version of that same question: Is it true that if $X$ is a regular ccc-space with a $G_\delta$-diagonal then $|X| \leq 2^{\omega}$ (see [2, p. 91, Question 16]). Shakhmatov answered their question in [22] by showing that there is no upper bound for the cardinality of completely regular ccc-spaces with $G_\delta$-diagonals. Then Arhangel’skii asked what if “$G_\delta$-diagonal” is replaced by “regular $G_\delta$-diagonal” [8]. Buzyakova answered that question by proving the following theorem:

**Theorem 1.1 ([8]).** The cardinality of a ccc-space with a regular $G_\delta$-diagonal does not exceed $2^\omega$.

In this paper we show that if $X$ is a Urysohn space then:
\[ |X| \leq 2^{e(X)\Delta(X)}; \]  
\[ |X| \leq 2^{\Delta(X)2^{wL(X)} }; \]  
\[ |X| \leq wL(X)\Delta(X)\chi(X); \]  
and
\[ |X| \leq aL(X)\Delta(X). \]  

Inequality (3) extends to the uncountable case Theorem 1.1. It follows from (4) that $2^{2^\omega}$ is an upper bound for the cardinality of weakly Lindelöf spaces with regular $G_\delta$-diagonals. This shows again that “regular $G_\delta$-diagonal” is a much more restrictive property than “$G_\delta$-diagonal”.
It follows from (5) that when $X$ is a space with a regular $G_\delta$-diagonal then $|X| \leq wL(X)^{\chi(X)}$. This improves significantly Bell, Ginsburg and Woods inequality for the class of normal spaces with regular $G_\delta$-diagonals. In particular (5) shows that the cardinality of every first countable space with a regular $G_\delta$-diagonal does not exceed $wL(X)^\omega$.

For the class of spaces with regular $G_\delta$-diagonals (6) improves Bella and Cammaroto inequality $|X| \leq 2^{\chi(X)}\cdot aL(X)$, which is valid for all Urysohn spaces [5]. Also, it follows from (6) that the cardinality of every space with a regular $G_\delta$-diagonal does not exceed $aL(X)^\omega$.

2. Definitions

Throughout this paper $\omega$ is (the cardinality of) the set of all non-negative integers, $\xi$ and $\eta$ are ordinals and $\tau$, $\mu$ and $\kappa$ are infinite cardinals. The cardinality of the set $X$ is denoted by $|X|$ and $\Delta_X = \{(x, x) \in X^2 : x \in X\}$ is the diagonal of $X$. If $\mathcal{U}$ is a family of subsets of $X$, $x \in X$, and $G \subseteq X$ then $st(G, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap G \neq \emptyset\}$. When $G = \{x\}$ we write $st(x, \mathcal{U})$ instead of $st(\{x\}, \mathcal{U})$. If $n \in \omega$, $st^n(G, \mathcal{U}) = st(st^{n-1}(G, \mathcal{U}), \mathcal{U})$ and $st^0(G, \mathcal{U}) = G$.

All spaces are assumed to be topological $T_1$-spaces. For a subset $U$ of a space $X$ the closure of $U$ (in $X$) is denoted by $\overline{U}$. $F \subseteq X$ is called regular-closed (in $X$) if there is open $U \subseteq X$ such that $F = \overline{U}$. As usual, $\chi(X)$ and $\psi(X)$ denote respectively the character and the pseudocharacter of $X$. The closed pseudo-character $\psi_c(X)$ (defined only for Hausdorff spaces $X$) is the smallest infinite cardinal $\kappa$ such that for each $x \in X$, there is a collection $\{V(x) : \eta < \kappa\}$ of open neighborhoods of $x$ such that $\bigcap_{\eta < \kappa} \overline{V}(\eta, x) = \{x\}$ [23]. The Hausdorff pseudo-character of $X$, denoted $H\psi(X)$, is the smallest infinite cardinal $\kappa$ such that for each $x \in X$, there is a collection $\{V(\eta, x) : \eta < \kappa\}$ of open neighborhoods of $x$ such that if $x \neq y$, then there exists $\eta, \xi < \kappa$ such that $V(\eta, x) \cap V(\xi, y) = \emptyset$ [17].

The Lindelöf number of $X$ is $L(X) = \min\{\kappa : \text{every open cover of } X \text{ has a subcover of cardinality } \leq \kappa\} + \omega$. The weak Lindelöf number of $X$, denoted $wL(X)$, is the smallest infinite cardinal $\kappa$ such that every open cover of $X$ has a subcollection of cardinality $\leq \kappa$ whose union is dense in $X$. If $wL(X) = \omega$ then $X$ is called weakly Lindelöf. The almost Lindelöf number of $X$, denoted $aL(X)$, is the smallest infinite cardinal $\kappa$ such that for every open cover $\mathcal{U}$ of $X$ there is a subcollection $\mathcal{U}_0$ such that $|\mathcal{U}_0| \leq \kappa$ and $\bigcup\{\mathcal{U} : \mathcal{U} \in \mathcal{U}_0\} = X$. If $aL(X) = \omega$ then $X$ is called almost Lindelöf. $e(X) = \sup\{|D| : D \subseteq X \text{ is closed and discrete}\} + \omega$ is the extent of $X$. A pairwise disjoint collection of non-empty open sets in $X$ is called a cellular family. The cellularity of $X$ is $c(X) = \sup\{|\mathcal{U}| : \mathcal{U} \text{ a cellular family in } X\} + \omega$. If $c(X) = \omega$ then it is called that $X$ satisfies the countable chain condition (or has the ccc) property.

A space $X$ has a $G_\kappa$-diagonal if there is a family $(U_\eta : \eta < \kappa)$ of open sets in $X^2$ such that $\Delta_X = \bigcap_{\eta < \kappa} U_\eta$; if $\Delta_X = \bigcap_{\eta < \kappa} U_\eta = \bigcap_{\eta < \kappa} \overline{U}_\eta$ then $X$ has a regular $G_\kappa$-diagonal. When $\kappa = \omega$ then $X$ has a $G_\delta$-diagonal (respectively,
regular $G_\delta$-diagonal). The diagonal degree of $X$, denoted $\Delta(X)$, is the smallest infinite cardinal $\kappa$ such that $X$ has a $G_\kappa$-diagonal (hence $\Delta(X) = \omega$ if and only if $X$ has a $G_\delta$-diagonal).

The following observation is well-known and easy to prove (see e.g. [10, Lemma 4.6]).

Lemma 2.1. $X$ has a diagonal which is the intersection of some of its regular-closed neighborhoods if and only if $X$ is a Urysohn space.

Definition 2.2. For a Urysohn space $X$ we define the regular diagonal degree $\Delta(X)$ of $X$ to be the minimal infinite cardinal $\kappa$ such that $X$ has a regular $G_\kappa$-diagonal.

Let $n$ be a positive integer. $X$ has a rank $n$-diagonal (a strong rank $n$-diagonal) if there is a sequence $\{U_m : m < \omega\}$ of open covers of $X$ such that for all $x \neq y$, there is some $m < \omega$ such that $y \notin \text{st}^n(x, U_m)$ ($y \notin \text{st}^n(x, U_m)$) ([3], [6]). Spaces $X$ with rank $n$-diagonals and strong rank $n$-diagonals were introduced and first studied in [18] under the names “spaces with $G_{\delta}(n)$-diagonals” and “spaces with $G_{\delta}(n)$-diagonals”. Clearly the spaces with strong rank 1-diagonals or, equivalently, the spaces with $G_{\delta}(1)$-diagonals, are exactly the spaces with $G_{\delta}^* \text{-diagonals}$ introduced and studied in [14].

The rank (strong rank) of the diagonal of a space $X$ is defined as the greatest natural number $n$ such that $X$ has a rank $n$-diagonal (strong rank $n$-diagonal), if such a number $n$ exists. The rank (strong rank) of the diagonal of $X$ is infinite, if $X$ has a rank $n$-diagonal (strong rank $n$-diagonal) for every $n \geq 1$ ([3], [6]).

Condensations are one-to-one and onto continuous mappings. A space $X$ is submetrizable if it condenses onto a metrizable space [3], or equivalently, $(X, \tau)$ is submetrizable if there exists a topology $\tau'$ on $X$ such that $\tau' \subset \tau$ and $(X, \tau')$ is metrizable [13].

A sequence $(U_n : n < \omega)$ of open covers of a space $X$ is a development for $X$ if for each $x \in X$, the set $\{\text{st}(x, U_n) : n < \omega\}$ is a base at $x$. A developable space is a space that has a development. A Moore space is a regular developable space [13].

For definitions not given here we refer the reader to [11], [19] or [15].

3. Preliminary results

The following three observations are well-known.

Lemma 3.1. For every Hausdorff space $X$

$$\psi(X) \leq \psi_c(X) \leq H\psi(X) \leq \chi(X).$$

Lemma 3.2. $w L(X) \leq a L(X) \leq L(X)$.

Lemma 3.3 ([15]). $w L(X) \leq c(X)$.

In 1961 Ceder made the following observation.
Lemma 3.4 ([9]). A space $X$ has a $G_\delta$-diagonal if and only if there is a sequence $(U_n : n < \omega)$ of open covers of $X$ such that if $x \in X$, then 

$$\{x\} = \cap_{n<\omega} \text{st}(x, U_n).$$

It is then clear that if a space $X$ has a $G_\delta$-diagonal then $\psi(X) = \omega$ and that $X$ has a $G_\delta$-diagonal if and only if $X$ has a rank 1-diagonal.

The case $\kappa = \omega$ of the following lemma was proved by Zenor in [24]. Below we show that Zenor’s proof works also for every $\kappa > \omega$.

Lemma 3.5. Let $\kappa$ be an infinite cardinal. A space $X$ has a regular $G_\kappa$-diagonal if and only if there is a family $(U_\eta : \eta < \kappa)$ of open covers of $X$ such that if $x$ and $y$ are distinct points of $X$, then there is $\eta < \kappa$ and open sets $U \in U_\eta$ and $V \in U_\eta$ containing $x$ and $y$ respectively, such that no member of $U_\eta$ intersects both $U$ and $V$.

Proof. Suppose that $X$ has a regular $G_\kappa$-diagonal and let $(W_\eta : \eta < \kappa)$ be a family of open sets in $X^2$ such that $\Delta_X = \cap_{\eta < \kappa} W_\eta = \cap_{\eta < \kappa} \bar{W}_\eta$. For each $\eta < \kappa$, let $U_\eta = \{U : U \in \text{open set of } X \text{ such that } U \times U \subset W_\eta\}$. To see that the family $(U_\eta : \eta < \kappa)$ is as required let $x$ and $y$ be a pair of distinct points of $X$. Then there exists $\eta < \kappa$ such that $(x, y)$ is not in $\bar{W}_\eta$ and open sets $U$ and $V$ in $X$, which contain $x$ and $y$ respectively, such that $U \times V$ does not intersect $W_\eta$, $U \times U \subset W_\eta$, and $V \times V \subset W_\eta$. To see that no member of $U_\eta$ intersects both $U$ and $V$, suppose otherwise; that is, suppose that $W$ is a member of $U_\eta$, $p$ is a point of $W \cap U$ and $q$ is a point of $W \cap V$. Then $(p, q)$ is a point of $W_\eta \cap (U \times V)$, which is a contradiction.

Now, suppose that $U_\eta$ is a family of open covers of $X$ as described in the lemma. For each $\eta < \kappa$, let $W_\eta = \cup_{\eta < \kappa} \{U \times U : U \in U_\eta\}$. Clearly, $\Delta_X \subset \cap_{\eta < \kappa} W_\eta$. To see that $\Delta_X = \cap_{\eta < \kappa} \bar{W}_\eta$, let $x$ and $y$ be distinct points of $X$. Then there is $\eta < \kappa$ and open sets $U, V \in U_\eta$ containing $x$ and $y$ respectively, such that no member of $U_\eta$ intersects both $U$ and $V$. It must be the case that $W_\eta$ does not intersect $U \times V$. \hfill \Box

In fact the proof of Lemma 3.5 shows a little bit more.

Corollary 3.6. If a space $X$ has a regular $G_\kappa$-diagonal, for some infinite cardinal $\kappa$, then there is a family $(U_\eta : \eta < \kappa)$ of open covers of $X$ such that

(a) if $x$ and $y$ are distinct points of $X$, then there exist $\eta < \kappa$ and open sets $U_\eta(x, y), U_\eta(y, x) \in U_\eta$ containing $x$ and $y$ respectively, such that $U_\eta(y, x) \cap \text{st}(U_\eta(x, y), U_\eta) = \emptyset$;

(b) if $x \in X$ then $\{x\} = \cap_{\eta < \kappa} \text{st}(x, U_\eta) = \cap_{\eta < \kappa} \overline{\text{st}(x, U_\eta)}$.

Proof. Let $(U_\eta : \eta < \kappa)$ be a sequence of open covers of $X$ as in Lemma 3.5.

(a) If $x, y \in X$ are distinct points then according to Lemma 3.5 there is $\eta < \kappa$ and open sets $U_\eta(x, y), U_\eta(y, x) \in U_\eta$ containing $x$ and $y$ respectively, such that no member of $U_\eta$ intersects both $U_\eta(x, y)$ and $U_\eta(y, x)$. Then $y \in U_\eta(y, x)$ and $U_\eta(y, x) \cap \text{st}(U_\eta(x, y), U_\eta) = \emptyset$. Therefore $U_\eta(y, x) \cap \text{st}(U_\eta(x, y), U_\eta) = \emptyset$.

(b) Follows immediately from (a). \hfill \Box
It follows from Corollary 3.6 that if a space $X$ has a regular $G_\kappa$-diagonal then $\psi_c(X) = \kappa$. In particular, if a space $X$ has a regular $G_\delta$-diagonal then $\psi_c(X) = \omega$.

Below we mention some well-known results and open questions closely related to spaces with regular $G_\delta$-diagonals.

**Lemma 3.7** ([3]). Every submetrizable space $X$ has a diagonal of infinite rank.

**Corollary 3.8** ([3], [6]). If the rank of the diagonal of a space $X$ is at least 3, then $X$ has a strong rank 2-diagonal.

**Corollary 3.9** ([3], [6]). If a space $X$ has a strong rank 2-diagonal, then $X$ has a regular $G_\delta$-diagonal.

**Corollary 3.10** ([13]). Every submetrizable space $X$ has a regular $G_\delta$-diagonal.

**Example 3.11** ([3, Example 2.9]). There exists a separable Tychonoff Moore space with a rank 3-diagonal (hence with a regular $G_\delta$-diagonal) that is not submetrizable.

**Lemma 3.12** ([3]). Every Moore space $X$ has a rank 2-diagonal.

**Lemma 3.13** ([13]). Not every Moore space $X$ has a regular $G_\delta$-diagonal.

**Question 3.14** (A. Bella (see [6], [3])). Is every regular $G_\delta$-diagonal a rank 2-diagonal?

As it is noted in [6], there is no example yet even of a space $X$ with a regular $G_\delta$-diagonal that does not have a strong rank 2-diagonal.

**Conjecture 3.15** ([3]). For every natural number $n$ there is a Tychonoff space $X_n$ with a rank $n$-diagonal that is not a rank $n + 1$-diagonal.

In 1991 Hodel established the following result.

**Theorem 3.16** ([16]). If $X$ is a Hausdorff space then $|X| \leq 2^{c(X)H\psi(X)}$.

A. Bella proved in [7] the following theorem:

**Theorem 3.17** ([7]). The cardinality of a ccc-space with a rank 2-diagonal does not exceed $2^\omega$.

Therefore if Question 3.14 has a positive answer then Buzyakova’s theorem (Theorem 1.1) will follow from Bella’s theorem (Theorem 3.17).

In [8] Buzyakova asked the following question (in that relation see Example 3.11):

**Question 3.18** ([8]). Is there a ccc-space with a regular $G_\delta$-diagonal that does not condense onto a first-countable Hausdorff space?

If the answer of the above question is in the negative then Buzyakova’s theorem will follow immediately from Hodel’s inequality (Theorem 3.16).
4. Main results

Theorem 4.2 below extends Buzyakova’s result (Theorem 1.1) for uncountable cardinalities. Its proof follows closely the original proof of Buzyakova. We begin first with a generalization for higher cardinalities of Lemma 2.1 from [8].

Lemma 4.1. Let $\kappa$ be an infinite cardinal, $X$ be a space with $c(X) = \kappa$ and $U \times V$ be a non-empty open set in $X^2$. Let also $U$ be a collection of open boxes in $X^2$ such that $U \times V \subseteq \bigcup U$. Then there exists $V = \{U_\eta \times V_\eta : \eta < \kappa\}$ such that $V \subseteq U$, $V \subseteq \bigcup_{\eta < \kappa} V_\eta$ and $U_\eta \times V_\eta$ meets $U \times V$ for each $\eta < \kappa$.

Proof. Let $U'$ consist of all elements of $U$ that meet $U \times V$. Since $U \times V \subseteq \bigcup U'$ and $U$ is not empty, we have $V \subseteq \bigcup\{V_\xi : U_\xi \times V_\xi \subseteq U'\}$. Since $c(X) = \kappa$ and $V$ is open in $X$, there exists $V = \{U_\eta \times V_\eta : \eta < \kappa\}$ such that $V \subseteq U'$, $V \subseteq \bigcup_{\eta < \kappa} V_\eta$ and $U_\eta \times V_\eta$ meets $U \times V$ for each $\eta < \kappa$. \hfill $\Box$

Theorem 4.2. If $X$ is a Urysohn space then $|X| \leq 2^{c(X) \cdot \Delta(X)}$.

Proof. Let $c(X) = \kappa$ and $\Delta(X) = \tau$. Then $X$ is a space with a regular $G_\tau$-diagonal. Let $\{W_\xi : \xi < \tau\}$ be a family of open sets in $X^2$ such that $\Delta_X = \bigcap\{W_\xi : \xi < \tau\} = \bigcap\{W_\xi : \xi < \tau\}$. For each $\xi < \tau$, fix a collection $\mathcal{U}_\xi$ of open boxes in $X^2 \setminus W_\xi$ such that $|\mathcal{U}_\xi| \leq 2^\kappa$ and $X^2 \setminus W_\xi \subseteq \bigcup \mathcal{U}_\xi$. Such a collection exists because $c(X^2) \leq 2^\kappa$ [20].

Now let $\mathcal{F}$ be a family of open sets in $X$ such that $F \in \mathcal{F}$ if and only if $F = X \setminus \bigcup\{V : U \times V \subseteq V \text{ for some } U\}$, where $V$ is such that $|V| \leq \kappa$ and $V \subseteq \mathcal{U}_\xi$ for some $\xi < \kappa$.

Since $|\mathcal{U}_\xi| \leq 2^\kappa$ for each $\xi < \tau$ and each $F \in \mathcal{F}$ is determined by a subset $\mathcal{V}$ of some $\mathcal{U}_\xi$ with cardinality at most $\kappa$, we have $|\mathcal{F}| \leq \tau \cdot 2^\kappa$. To finish the proof that $|X| \leq 2^{\kappa \cdot \tau}$ it is sufficient to show that for every $x \in X$ there exists a family $\{F_\xi : \xi < \tau\} \subseteq \mathcal{F}$ such that $\{x\} = \bigcap\{F_\xi : \xi < \tau\}$.

Let $x \in X$ be fixed. For each $\xi < \tau$ we shall define $F_\xi$. Let $B$ be an open set in $X$ such that $x \in B$ and $B \times B \subseteq W_\xi$. Fix a collection $\mathcal{O}$ of open boxes in $X^2 \setminus W_\xi$ with $|\mathcal{O}| \leq \kappa$ and such that:

(a) $x \in U \subseteq B$ for every $U \times V \subseteq \mathcal{O}$; and
(b) $\bigcup\{V : U \times V \subseteq \mathcal{O} \text{ for some } U\}$ contains $\{y : (x, y) \notin W_\xi\}$.

In order to construct such a collection we first cover $\{x\} \times X \setminus W_\xi$ by open boxes in $X^2 \setminus W_\xi$ satisfying (a) and then using the fact that $c(X) \leq \kappa$ we find the desired subcollection $\mathcal{O}$.

For each $U \times V \subseteq \mathcal{O}$, fix $\mathcal{V}_{U \times V} = \{U_\eta \times V_\eta : \eta < \kappa\}$ such that $\mathcal{V}_{U \times V} \subseteq \mathcal{U}_\xi$ satisfies the conclusion of Lemma 4.1 and let $G_{U \times V} = \bigcup \mathcal{V}_{U \times V} = \{V_\eta : \eta < \kappa\}$.

Now we shall verify the following two observations:

(1) $V_\eta \cap B = \emptyset$ for each $\eta < \kappa$. Indeed, by (a) $U \subseteq B$ and by the conclusion of Lemma 4.1, $U \cap \mathcal{U}_\eta \neq \emptyset$. Hence $U_\eta \cap B \neq \emptyset$. Now suppose that $V_\eta \cap B \neq \emptyset$. Then $(U_\eta \times V_\eta) \cap (B \times B) \neq \emptyset$. However $B \times B \subseteq W_\xi$ while $U_\eta \times V_\eta \in \mathcal{U}_\xi$ is a subset of $X^2 \setminus W_\xi$ – contradiction.
Take infinite cardinals $\kappa$, $\lambda$, and $\mu$ such that $\Delta(X) \leq \kappa$, $wL(X) \leq \lambda$, and $\chi(X) \leq \mu$. Then, according to Lemma 3.5, there is a family $\{U_\eta : \eta < \kappa\}$ of open covers of $X$ such that for any distinct points $x, y \in X$ we can find an ordinal $\eta = \eta(x,y) < \kappa$ and a set $U_\eta(x,y) \in U_\eta$ such that $x \in U_\eta(x,y)$ and $y \notin \overline{st(U_\eta(x,y),U_\eta)}$. Let $Y_\eta(x) = \{y : y \in X \setminus \{x\}, \eta(x,y) = \eta\}$. For $y \in Y_\eta(x)$ let $D^x_\eta(y) = \{U : U \in D_\eta, U \cap U_\eta(y,x) \neq \emptyset\}$. Then $U_\eta(x,y) \cap \bigcup D^x_\eta(y) = \emptyset$, hence $x \notin \bigcup D^x_\eta(y)$. Therefore $\{x\} = \bigcap\{X \setminus \bigcup D^x_\eta(y) : y \in Y_\eta(x)\} : \eta < \kappa$.

For each $x \in X$ and $\eta < \kappa$, there are at most $2^\tau$ sets of the form $D^x_\eta(y)$ for $|D_\eta| \leq \tau$. Hence, there are at most $2^{2^\tau}$ sets of the form $\bigcap\{X \setminus \bigcup D^x_\eta(y) : y \in Y_\eta(x)\}$. Thus, there are at most $2^{2^{2^\tau}}$ many intersections of the form $\bigcap\{X \setminus \bigcup D^x_\eta(y) : y \in Y_\eta(x)\} : \eta < \kappa$. Therefore we conclude that $|X| \leq 2^{2^\aleph_0}$.

**Theorem 4.6.** Let $X$ be a Urysohn space. Then $|X| \leq wL(X)\chi(X)\Delta(X)$.

**Proof.** Take infinite cardinals $\kappa$, $\lambda$, and $\mu$ such that $\Delta(X) \leq \kappa$, $wL(X) \leq \lambda$, and $\chi(X) \leq \mu$. Then, according to Lemma 3.5, there is a family $\{U_\eta : \eta < \kappa\}$ of open covers of $X$ such that for any distinct points $x, y \in X$ we can find an ordinal $\eta = \eta(x,y) < \kappa$ and a set $U_\eta(x,y) \in U_\eta$ such that $x \in U_\eta(x,y)$ and $y \notin \overline{st(U_\eta(x,y),U_\eta)}$. Let $Y_\eta(x) = \{y : y \in X \setminus \{x\}, \eta(x,y) = \eta\}$. Notice
that from \(\omega L(X) \leq \lambda\) it follows that for any \(\eta < \kappa\) there exists a family \(D_\eta \subset U_\eta\) such that \(|D_\eta| \leq \lambda\) and \(\bigcup D_\eta = X\).

For every \(x \in X\) fix a base \(V_x = \{V_\xi(x) : \xi < \mu\}\) of \(X\) at the point \(x\). Given \(\eta < \kappa\) and \(\xi < \mu\) fix \(D_\xi(x) \in D_\eta\) such that \(D_\xi(x) \cap V_\xi(x) \neq \emptyset\); observe that such an element of \(D_\eta\) exists because the union of \(D_\eta\) is dense in \(X\).

Let \(D_\eta(x) = \{D_\xi(x) : \xi < \mu\}\) and for each \(\xi < \mu\) let \(D_\eta(x, \xi) = \{D : D \in D_\eta(x), D \cap V_\xi(x) \neq \emptyset\}\). It is clear that for each \(\xi < \mu\) we have \(D_\eta(x, \xi) \neq \emptyset\) and \(|D_\eta(x, \xi)| \leq \mu\). We claim that \(x \in \bigcup D_\eta(x, \xi)\). To see that any open neighborhood \(O\) of \(x\). Then \(O \cap V_\xi(x)\) is also an open neighborhood of \(x\) and therefore there is \(\xi' < \mu\) such that \(V_{\xi'}(x) \subset O \cap V_\xi(x)\). Thus, \(D_{\xi'}(x) \in D_\eta(x, \xi)\). Hence \(O \cap D_{\xi'}(x) \neq \emptyset\) and therefore \(O \cap (\bigcup D_\eta(x, \xi)) \neq \emptyset\).

Notice that for every \(y \in Y_\eta(x)\) there is \(\xi < \mu\) such that \(V_\xi(x) \subset U_\eta(x, y)\) and hence \(y \notin \bigcup D_\eta(x, \xi)\). Thus, for each \(\eta < \kappa\) we have \(x \in \bigcap \bigcup D_\eta(x, \xi)\) : \(\xi < \mu\) and \(V_\eta(x) \cap (\bigcap \bigcup D_\eta(x, \xi) : \xi < \mu) = \emptyset\). Therefore \([x] = \bigcap \{\bigcap \bigcup D_\eta(x, \xi) : \xi < \mu\} : \eta < \kappa\}.

For each \(\eta < \kappa\) there are at most \(\lambda^\eta\) possible subsets of \(D_\eta\) of the type \(D_\eta(x)\). For each such set, there are at most \(2^\mu\) subsets of the type \(D_\eta(x, \xi)\). Hence, there are at most \(\lambda^\mu \cdot 2^\mu = \lambda^\mu\) possible subsets of \(D_\eta\) of the type \(D_\eta(x, \xi)\). Thus, for each \(\eta < \kappa\) there are at most \((\lambda^\mu)^\mu = \lambda^\mu\) possible intersections \(\bigcap \{\bigcup D_\eta(x, \xi) : \xi < \mu\}\). Therefore there are at most \((\lambda^\mu)^\kappa\) possible intersections \(\bigcap \{\bigcap \{\bigcup D_\eta(x, \xi) : \xi < \mu\} : \eta < \kappa\}\), i.e., we proved that \(|X| \leq \lambda^{\kappa^\mu}\).

**Corollary 4.7.** If \(X\) is a Urysohn space with a regular \(G_\delta\)-diagonal then \(|X| \leq \omega L(X)\chi(X)\).

**Corollary 4.8.** Let \(X\) be a first countable Urysohn space with a regular \(G_\delta\)-diagonal. Then \(|X| \leq \omega L(X)^\omega\).

**Corollary 4.9.** If \(X\) is a weakly Lindelöf, first countable, Urysohn space then \(|X| \leq 2^{\chi(X)}\).

**Corollary 4.10.** Let \(X\) be a weakly Lindelöf, developable, Urysohn space. Then \(|X| \leq 2^{\chi(X)}\).

**Corollary 4.11.** The cardinality of every weakly Lindelöf, first countable, Urysohn space with a regular \(G_\delta\)-diagonal does not exceed \(2^\omega\).

**Corollary 4.12.** Let \(X\) be a weakly Lindelöf, Moore space. If \(|X| > 2^\omega\) then \(X\) is not submetrizable.

We note that it follows from Corollary 4.8 that if \(X\) is a developable, Urysohn space with a regular \(G_\delta\)-diagonal then \(|X| \leq \omega L(X)^\omega\). Also, in the proof of [24, Theorem 2] among other things Zenor showed that the following lemma is true.

**Lemma 4.13.** If \(X\) is a developable Urysohn space then \(X\) has a regular \(G_\delta\)-diagonal if and only if \(X\) has a rank 3-diagonal.
Therefore for the class of developable Urysohn spaces Corollary 4.8 is equivalent to Proposition 4.7 from [6]: If \( X \) has a rank 3-diagonal then \(|X| \leq wL(X)^\omega\). In relation to that result, Theorem 4.6 and Corollary 4.12 the following questions are of interest:

**Question 4.14** ([6]). Is it the case that if \( X \) has a strong rank 2-diagonal then \(|X| \leq wL(X)^\omega\)?

**Question 4.15.** Is there a weakly Lindelöf, Moore space with cardinality greater than \(2^\omega\)?

We recall that in [21] the author attributed to [24] the following result: A \(T_2\)-space \( X \) has a regular \(G_\delta\)-diagonal if and only if it has a rank 3-diagonal. (We cite that result using the current terminology. For the exact statement see [21, Theorem 5]). But, since there is no proof of this claim in [24], and in relation to Questions 3.14 and 4.14 and some of the above results we believe that the author of [21] had in mind Lemma 4.13 instead.

We finish with a theorem that gives an upper bound for the cardinality of a Urysohn space \( X \) as a function of \(aL(X)\) and \(\overline{\Delta}(X)\).

**Theorem 4.16.** If \( X \) is a Urysohn space then \(|X| \leq aL(X)\overline{\Delta}(X)\).

**Proof.** Let \(\overline{\Delta}(X) = \kappa\). Then \( X \) is a space with a regular \(G_\kappa\)-diagonal. Therefore, according to Corollary 3.6, there is a family \((\mathcal{U}_\eta : \eta < \kappa)\) of open covers of \( X \) such that if \( x \) and \( y \) are distinct points of \( X \), then there is \( \eta < \kappa \) and an open set \( U_\eta(x,y) \in \mathcal{U}_\eta \) containing \( x \) such that \( y \notin \text{st}(U_\eta(x,y),U_\eta)\).

Let \( aL(X) = \tau \). Then for each \( \eta < \kappa \) we can find a subfamily \( \mathcal{D}_\eta \) of \( \mathcal{U}_\eta \) such that \(|\mathcal{D}_\eta| \leq \tau \) and \( X = \bigcup(\mathcal{D} : D \in \mathcal{D}_\eta)\).

Let \( x \in X \). For each \( \eta < \kappa \) we fix \( D_{x,\eta} \in \mathcal{D}_\eta \) such that \( x \in \overline{D_{x,\eta}} \).

Now, let \( y \in X \setminus \{x\} \). Then there is \( \eta < \kappa \) and an open set \( U_\eta(x,y) \in \mathcal{U}_\eta \) containing \( x \) such that \( y \notin \text{st}(U_\eta(x,y),U_\eta)\). Since \( D_{x,\eta} \in \mathcal{D}_\eta \subset \mathcal{U}_\eta \), we have \( D_{x,\eta} \subset \text{st}(U_\eta(x,y),U_\eta) \). Hence \( D_{x,\eta} \subset \text{st}(U_\eta(x,y),U_\eta) \) and therefore \( y \notin \overline{D_{x,\eta}} \). This shows that \( \{x\} = \bigcap_{\eta < \kappa} \overline{D_{x,\eta}} \).

Since each \( D_{x,\eta} \) could be chosen out of \( \tau \) many sets, there are \( \tau^\kappa \) such possible intersections. Therefore we conclude that \(|X| \leq \tau^\kappa\). \[ \square \]

**Corollary 4.17.** If \( X \) is a Urysohn space with a regular \(G_\delta\)-diagonal then \(|X| \leq aL(X)^\omega\).

**Corollary 4.18.** The cardinality of every almost Lindelöf Urysohn space with a regular \(G_\delta\)-diagonal does not exceed \(2^\omega\).

Notice that the results in Corollary 4.17 and in Corollary 4.8 strengthen, for the class of spaces with regular \(G_\delta\)-diagonals, the following Bella and Cammaroto result: Let \( X \) be a Urysohn space. Then \(|X| \leq 2^{\chi(X)\cdot aL(X)} \) [5].

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REFERENCES


Department of Mathematical Sciences, Central Connecticut State University, 1615 Stanley Street, New Britain, CT 06050
E-mail address: gotchevi@ccsu.edu