THE NON-HAUSDORFF NUMBER OF A TOPOLOGICAL SPACE

IVAN S. GOTCHEV

Abstract. We call a non-empty subset $A$ of a topological space $X$ finitely non-Hausdorff if for every non-empty finite subset $F$ of $A$ and every family $\{U_x : x \in F\}$ of open neighborhoods $U_x$ of $x \in F$, $\bigcap \{U_x : x \in F\} \neq \emptyset$ and we define the non-Hausdorff number $nh(X)$ of $X$ as follows: $nh(X) := 1 + \sup \{|A| : A$ is a finitely non-Hausdorff subset of $X\}$.

Using this new cardinal function we show that the following three inequalities are true for every topological space $X$:

(i) $|X| \leq 2^{2^{d(X)} \cdot nh(X)}$;
(ii) $w(X) \leq 2^{2^{w(X)} \cdot nh(X)}$;
(iii) $|X| \leq d(X)^{\chi(X)} \cdot nh(X)$

and

(iv) $|X| \leq nh(X)^{\chi(X)L(X)}$

is true for every $T_1$-topological space $X$, where $d(X)$ is the density, $w(X)$ is the weight, $\chi(X)$ is the character and $L(X)$ is the Lindelöf degree of the space $X$.

The first three inequalities extend to the class of all topological spaces Pospišil’s inequalities that for every Hausdorff space $X$, $|X| \leq 2^{2^{d(X)}}$, $w(X) \leq 2^{2^{w(X)}}$ and $|X| \leq d(X)^{\chi(X)}$. The fourth inequality generalizes to the class of all $T_1$-spaces Arhangel’skii’s inequality that for every Hausdorff space $X$, $|X| \leq 2^{\chi(X)L(X)}$. It is still an open question if Arhangel’skii’s inequality is true for all $T_1$-spaces. It follows from (iv) that the answer of this question is in the affirmative for all $T_1$-spaces with $nh(X)$ not greater than the cardinality of the continuum.

Examples are given to show that the upper bounds in (i) and (iii) are exact and that $nh(X)$ cannot be omitted.

1. Introduction

Let $X$ be a topological space and for $x \in X$ let $N_x$ denote the family of all open neighborhoods of $x$ in $X$. For a nonempty subset $A$ of $X$ we denote by $U_A$ the set of all families $U := \{U_a : a \in A, U_a \in N_a\}$. A family $\Gamma$ is centered if the intersection of any finitely many elements of $\Gamma$ is non-empty.

2010 Mathematics Subject Classification. Primary 54A25, 54D10.
Key words and phrases. Cardinal function, the non-Hausdorff number of a space, (maximal) non-Hausdorff subset of a space, Arhangel’skii’s inequality, Pospišil’s inequalities.
This paper is respectfully dedicated to W. W. Comfort on the occasion of his 80th birthday.
Recall that $d(X) := \min\{|A| : A \subset X, A = X\}$, $w(X) := \min\{|B| : B$ is a base for $X\}$, $L(X) := \min\{\kappa : \text{every open cover of } X \text{ has a subcover of cardinality } \leq \kappa\}$, $\chi(x, X) := \min\{|V| : V$ is a local base for $x\}$ and $\chi(X) := \sup\{\chi(x, X) : x \in X\}$.

In 1937, Pospišil proved (see [9]) that for every Hausdorff space $X$,

(a) $|X| \leq 2^{2d(X)}$;
(b) $w(X) \leq 2^{2^{2d(X)}}$; and
(c) $|X| \leq d(X)\chi(X)$;

and in 1969, Arhangel’skiĭ (see [1]) answered a question of Alexandroff and Urysohn raised in 1923 by showing that for every Hausdorff space $X$, $|X| \leq 2^{\chi(X)L(X)}$. Since then many mathematicians have obtained similar inequalities for different classes of topological spaces but it is still unknown if Arhangel’skiĭ’s inequality is true for all $T_1$-topological spaces (see the survey paper [8]).

In this paper we generalize Pospišil’s inequalities for the class of all topological spaces and Arhangel’skiĭ’s inequality for the class of all $T_1$-topological spaces and show that Arhangel’skiĭ’s inequality is true for a very large class of $T_1$-spaces.

2. The cardinal function $nh(X)$

We begin with an example showing that Pospišil’s inequality (c) is not always true for $T_1$-spaces.

**Example 2.1.** Let $\mathbb{N}$ denote the set of all positive integers and $\mathbb{R}$ be the set of all real numbers. Let $S := \{1/n : n \in \mathbb{N}\}$ and $M := S \cup \{0\}$ be the subspace of $\mathbb{R}$ with the inherited topology. Then in $M$ all points except 0 are isolated and $\lim_{n \to \infty} 1/n = 0$. Let $\alpha$ be an initial ordinal. We duplicate $\alpha$ many times the point 0 $\in M$ i.e. we replace in $M$ the point 0 with $\alpha$ many distinct points and obtain the set $X := S \cup \alpha$ with topology such that for each $\beta < \alpha$ we have $\lim_{n \to \infty} 1/n = \beta$ and the subspaces $S$ and $\alpha$ with the inherited topology from $X$ are discrete. Then the set $\{1/n : n \in \mathbb{N}\}$ is dense in $X$ (hence $d(X) = \omega$), $\chi(X) = \omega$, and if $\alpha > 2^\omega$ then $|X| > d(X)\chi(X) = \omega^\omega = 2^\omega$.

To be able to generalize Pospišil’s and Arhangel’skiĭ’s inequalities we need to introduce some new concepts.

**Definition 2.2.** We will call a nonempty subset $A$ of a topological space $X$ finitely non-Hausdorff if for every non-empty finite subset $F$ of $A$ and every $U \in \mathcal{U}_F$, $\cap U \neq \emptyset$. The set $A$ will be called maximal finitely non-Hausdorff subset of $X$ if $A$ is a finitely non-Hausdorff subset of $X$ and if $B$ is a finitely non-Hausdorff subset of $X$ such that $A \subset B$ then $A = B$.

We note that in a Hausdorff space $X$ the only maximal finitely non-Hausdorff subsets of $X$ are the singletons.

**Lemma 2.3.** Every finitely non-Hausdorff subset of a topological space $X$ is contained in a maximal finitely non-Hausdorff subset of $X$. 

Proof. It is a direct corollary of Zorn’s lemma.

Now we are ready to introduce the concept of a non-Hausdorff number of a topological space $X$.

**Definition 2.4.** Let $X$ be a topological space. We define the non-Hausdorff number $nh(X)$ of $X$ as follows: $nh(X) := 1 + \sup\{|A| : A \text{ is a (maximal) finitely non-Hausdorff subset of } X\}$.

**Remark 2.5.** It follows from Definition 2.4 that $X$ is a T$_2$-space if and only if $nh(X) = 2$ and $2 < nh(X) \leq 1 + |X|$ whenever $X$ is a non-Hausdorff space. Also, if $X$ is a topological space and $A \subset X$ then $nh(A) \leq nh(X)$, and if $X$ is an infinite set with topology generated by the open sets $\{X \setminus \{x\} : x \in X\}$ then $X$ is a maximal finitely non-Hausdorff set and therefore $nh(X) = |X|$. Finally, using similar ideas as in Example 2.1 one can construct T$_1$-spaces $X$ with one or more of the following properties:

(i) there exist maximal finitely non-Hausdorff subsets $M$ and $N$ of $X$ such that $|M \cap N| \geq 0$ and $|M|, |N|, |M \cap N|$ could have any cardinality which satisfy $0 \leq |M \cap N| \leq |M|$ and $0 \leq |M \cap N| \leq |N|$;

(ii) there exists a maximal finitely non-Hausdorff subset $M$ and a point $x \in M$ such that $M \subseteq \bigcap\{U_x : U_x \in \mathcal{N}_x\}$ and $|M| = nh(X)$.

We finish this section with two observations about finitely non-Hausdorff subsets of topological spaces.

**Lemma 2.6.** Let $X$ be a topological space and $A$ be a finitely non-Hausdorff subset of $X$. Then $A \subset \bigcap\{\overline{U} : U \in \mathcal{U}_F, \emptyset \neq F \subset A, |F| < \omega\}$.

**Proof.** Let $F$ be a nonempty subset of $A$, $U_0 \in \mathcal{U}_F$, and $G = \cap U_0$. Suppose that there exist $a_0 \in A$ such that $a_0 \notin G$. Then there is $W_{a_0} \in \mathcal{N}_{a_0}$ such that $W_{a_0} \cap G = \emptyset$. Let $V_{a_0} = W_{a_0}$ if $a_0 \notin F$ and $V_{a_0} = U_0 \cap W_{a_0}$, where $U_{a_0} \in U_0$ and $U_{a_0} \in \mathcal{N}_{a_0}$, if $a_0 \in F$. Then the family $\mathcal{U}_1 := \{V_{a_0}\} \cup \{U_a : U_a \in U_0, a \notin F\}$ has the property that $\bigcap \mathcal{U}_1 = \emptyset$ - a contradiction. Therefore $A \subset \overline{U}$ for every $U \in \mathcal{U}_F$ and every nonempty subset $F$ of $A$ with $|F| < \omega$. Thus $A \subset \bigcap\{\overline{U} : U \in \mathcal{U}_F, \emptyset \neq F \subset A, |F| < \omega\}$. □

**Theorem 2.7.** Let $X$ be a topological space and $A$ be a maximally finitely non-Hausdorff subset of $X$. Then $A = \bigcap\{\overline{U} : U \in \mathcal{U}_F, \emptyset \neq F \subset A, |F| < \omega\}$.

**Proof.** Let $A$ be a maximal finitely non-Hausdorff subset of $X$. Then it follows from Lemma 2.6 that $A \subset \bigcap\{\overline{U} : U \in \mathcal{U}_F, \emptyset \neq F \subset A, |F| < \omega\}$. Suppose that there is $x_0 \in \bigcap\{\overline{U} : U \in \mathcal{U}_F, \emptyset \neq F \subset A, |F| < \omega\}$ \setminus $A$. Then $U \cap (\overline{U}) \neq \emptyset$ for every $U \in \mathcal{N}_{x_0}$, every $U \in \mathcal{U}_F$ and every nonempty finite subset $F$ of $A$. Thus for the set $A_1 := A \cup \{x_0\}$ we have that if $F \subset A_1$ with $F \neq \emptyset$ and $|F| < \omega$, and $U \in \mathcal{U}_F$ then $\cap U \neq \emptyset$. Therefore, $A_1$ is a finitely non-Hausdorff subset of $X$ and $A \subseteq A_1$ - a contradiction with the maximality of $A$. □
3. Some Cardinal Inequalities Involving the Non-Hausdorff Number

We begin with the generalization of Pospíšil’s inequalities (a), (b) and (c) for the class of all topological spaces.

**Theorem 3.1.** Let $X$ be a topological space. Then $|X| \leq 2^{2^{w(X)}} \cdot nh(X)$. 

**Proof.** Let $D$ be a dense subset of $X$ with $d(X) = |D|$ and $u = hn(X)$. For every nonempty finite subset $A$ of $X$ and every point $x \in A$ we choose a subset $G_{A,x}$ of $D$ with $x \in G_{A,x}$ in the following way:

(i) If $A$ is a finitely non-Hausdorff subset of $X$ then $G_{A,x} := D$; and

(ii) If $A$ is not a finitely non-Hausdorff subset of $X$ then for each $x \in A$ we choose $U_{A,x} \in \mathcal{N}_x$ such that $\cap\{U_{A,x} : x \in A\} = \emptyset$. Then for $x \in A$ we let $G_{A,x} := U_{A,x} \cap D$. 

Now, for $x \in X$ let $\Gamma_x := \{G_{A,x} : A \subset X, \emptyset \neq |A| < \omega, x \in A\}$. Then, for each $x \in X$, $\Gamma_x$ is a centered family and $\Gamma_x \in \mathcal{P}(\mathcal{P}(D))$. We claim that the mapping $x \rightarrow \Gamma_x$ from $X$ to $\mathcal{P}(\mathcal{P}(D))$ is $(\leq u)$-to-one. Assume the contrary. Then there is a subset $K \subset X$ such that $|K| = u^+$ and every $x \in K$ corresponds to the same centered family $\Gamma$. Since $hn(X) = u$, $K$ is not a finitely non-Hausdorff subset of $X$. Then there exists $F \subset K$ with $\emptyset \neq |F| < \omega$ and $U \in \mathcal{U}_F$ such that $\cap U = \emptyset$. Then for each $x \in F$ we have $\cap U \in \Gamma$; hence $\Gamma$ is not centered - a contradiction. Therefore $|X| \leq 2^{2^{w(D)}} \cdot u$. □

**Corollary 3.2.** Let $X$ be a topological space. Then $w(X) \leq 2^{2^{(w(X))} \cdot nh(X)}$. 

**Proof.** It follows directly from Theorem 3.1 and the fact that for any topological space $X$, $w(X) \leq 2^{|X|}$ (see [7, 3.1a]). □

**Remark 3.3.** Example 2.1 shows that the upper bound in the inequality in Theorem 3.1 is exact. To see that, let $\alpha > 2^{2^\omega}$. Then $nh(X) = \alpha$ and $\alpha = |X| \leq 2^{2^\omega} \cdot nh(X) = \alpha$.

**Theorem 3.4.** Let $A$ be a subset of a topological space $X$. Then $|A| \leq |A|^{\chi(\overline{A})} \cdot nh(\overline{A})$. 

**Proof.** Let $\chi(\overline{A}) = m$, $nh(\overline{A}) = u$, and $|A| = \tau$. For each $x \in \overline{A}$ let $\mathcal{V}_x$ be a local base for $x$ in $\overline{A}$ with $|\mathcal{V}_x| \leq m$. For every $x \in \overline{A}$ and every $V \in \mathcal{V}_x$, fix a point $a_{x,V} \in V \cap A$, and let $A_x := \{a_{x,V} : V \in \mathcal{V}_x\}$. Let also $\Gamma_x := \{V \cap A_x : V \in \mathcal{V}_x\}$. Then $\Gamma_x$ is a centered family. It is not difficult to see that there are at most $\tau^m$ such centered families. Indeed $A_x \in [A]^{\leq m}$ and $V \cap A_x \in [A]^{\leq m}$, for every $V \in \mathcal{V}_x$. Since $|\mathcal{V}_x| \leq m$, each centered family $\Gamma_x$ is an element of $[\tau]^m \leq m$ and therefore there are at most $m = |A|^m = \tau^m$ such families.

We claim that the mapping $x \rightarrow \Gamma_x$ is $(\leq u)$-to-one. Assume the contrary. Then there is a subset $K \subset \overline{A}$ such that $|K| = u^+$ and every $x \in K$ corresponds to the same centered family $\Gamma$. Since $nh(\overline{A}) = u$, $K$ is not a
Clearly defined. We will define \( F \) as the finitely non-Hausdorff subset of \( A \). Then there exist \( F \subseteq K \) with \( \emptyset \neq |F| < \omega \) and \( U \subseteq U_F \) such that \( \cap U = \emptyset \). Hence for every \( x \in F \) and \( U_x \in U \) we have \( U_x \cap A_x \in \Gamma \); thus \( \Gamma \) is not centered - a contradiction.

Therefore the mapping \( x \to \Gamma_x \) from \( A \) to \([A]^{\leq m}]^{\leq m} \) is \((e)\) -to-one, and thus
\[
|A| \leq u \cdot (\tau^m)^m = u \cdot m^m
\]

\( \square \)

**Corollary 3.5.** Let \( A \) be a subset of a topological space \( X \). Then \( |A| \leq |A|^{\chi(X)} \cdot nh(X) \).

**Corollary 3.6.** Let \( X \) be a topological space. Then \( |X| \leq d(X)^{\chi(X)} \cdot nh(X) \).

**Remark 3.7.** Example 2.1 shows that the upper bound in the inequality in Corollary 3.6 (and Theorem 3.1) is exact. To see that, let \( \alpha > 2^{2^{\omega}} \). Then \( nh(X) = \alpha \) and \( \alpha = |X| \leq d(X)^{\chi(X)} \cdot nh(X) = \omega \cdot \alpha \).

The following theorem generalizes Arhangel’skii’s inequality for the class of \( T_1 \)-topological spaces.

**Theorem 3.8.** For every \( T_1 \)-topological space \( X \), \( |X| \leq nh(X)^{\chi(X)} \cdot nh(X) \).

**Proof.** Let \( \chi(X)L(X) = m \) and \( nh(X) = u \). For each \( x \in X \) let \( V_x \) be a local base for \( x \) with \( |V_x| \leq m \). Let \( x_0 \) be an arbitrary point in \( X \). Recursively we construct a family \( \{F_\alpha : \alpha < m^+\} \) of subsets of \( X \) with the following properties:

(i) \( F_0 = \{x_0\} \) and \( \bigcup_{\beta < \alpha} F_\beta \subseteq F_\alpha \) for every \( 0 < \alpha < m^+ \);
(ii) \( |F_\alpha| \leq u^m \) for every \( \alpha < m^+ \);
(iii) for every \( \alpha < m^+ \), and every \( F \subseteq \bigcup_{\beta < \alpha} F_\beta \) with \( |F| \leq m \) if \( X \setminus U \neq \emptyset \).

Suppose that the sets \( \{F_\beta : \beta < \alpha\} \) satisfying (i)-(iii) have already been defined. We will define \( F_\alpha \). Since \( |F_\beta| \leq u^m \) for each \( \beta < \alpha \), we have \( \bigcup_{\beta < \alpha} F_\beta \) \( \leq u^m \cdot m^+ = u^m \). Then it follows from Corollary 3.5, that \( |\bigcup_{\beta < \alpha} F_\alpha| \leq u^m \). Therefore there are at most \( u^m \) subsets \( F' \subseteq \bigcup_{\beta < \alpha} F_\alpha \) with \( |F'| \leq m \) and for each such set \( F' \) we have \( |U_F| \leq m^m = 2^m \leq u^m \). For each \( F \subseteq \bigcup_{\beta < \alpha} F_\alpha \) with \( |F| \leq m \) and each \( U \in U_F \) for which \( X \setminus U \neq \emptyset \) we choose a point in \( X \setminus \cup U \neq \emptyset \) and let \( E_\alpha \) be the set of all these points. Clearly \( |E_\alpha| \leq u^m \). Let \( F_\alpha = F_\alpha \cup \bigcup_{\beta < \alpha} F_\beta \). Then it follows from our construction that \( F_\alpha \) satisfies (i) and (iii) while (ii) follows from Corollary 3.5.

Now let \( G = \bigcup_{\alpha < m} F_\alpha \). Clearly \( |G| \leq u^m \cdot m^+ = u^m \). We will show that \( G \) is closed. Suppose the contrary and let \( x \in \bar{G} \setminus G \). Then for each \( U \in V_x \) we have \( U \cap G \neq \emptyset \) and therefore there is \( \alpha_U < m^+ \) such that \( U \cap F_{\alpha_U} \neq \emptyset \). Since \( |\{U : U \in V_x\}| \leq m \), there is \( \beta < m^+ \) such that \( \beta > \alpha_U \) for every \( U \in V_x \) and therefore \( x \in F_\beta \subseteq G \) - a contradiction.

To finish the proof it remains to check that \( G = X \). Suppose that there is \( x \in X \setminus G \). Since \( X \) is \( T_1 \), for every \( y \in G \) there is \( V_y \in V \) such that
$x \notin V_y$. Then $\{X \setminus G\} \cup \{V_y : y \in G\}$ is an open cover of $X$. Thus there exists $F \subseteq G$ with $|F| \leq m$ such that $G \subseteq \cup_{y \in F} V_y$. Since $|F| \leq m$, there is $\beta < m^+$ such that $F \subseteq F_\beta$. Then for $U := \{V_y : y \in F\}$ we have $U \in \mathcal{U}_F$ and $x \in X \setminus \bigcup U$. Therefore $F_{\beta + 1} \setminus \bigcup U \neq \emptyset$ - a contradiction. \hfill \Box

**Corollary 3.9.** Let $X$ be an infinite $T_1$-topological space with $\text{nh}(X)$ not greater than the cardinality of the continuum. Then $|X| \leq 2^{\chi(X)L(X)}$.

**Proof.** For every infinite $T_1$-topological space $X$ either $\chi(X)$ or $L(X)$ is infinite. \hfill \Box

**Example 3.10.** Let $I$ be the unit interval with its standard topology and $Y$ be a set with cardinality $\alpha > 2^{2^\omega}$. Let $X := I \cup Y$ be the topological space with the following topology: every point $y \in Y$ is an open set but not a closed set in $X$ and $U \subseteq X$ is a neighborhood of a point $i \in I$ if and only if $U = V \cup Y$ where $V \subseteq I$ is a standard neighborhood of $i$ in $I$. Then $X$ is a $T_0$ topological space with $\chi(X) = L(X) = \omega$ and $\text{nh}(X) = 2^\omega$ but $|X| = \alpha > (2^\omega)^{2^\omega} = 2^{2^\omega}$. Therefore Theorem 3.8 is not necessarily valid for $T_0$-topological spaces.

**Remark 3.11.** For results about cardinal inequalities for topological spaces involving the Hausdorff number of a topological space see [2]; for results involving the Urysohn number of a space see [4], [3] and [5]; and for results involving the non-Urysohn number of a topological space see [6].

**References**


Department of Mathematical Sciences, Central Connecticut State University, 1615 Stanley Street, New Britain, CT 06050
E-mail address: gotchevi@ccsu.edu