TOPOLOGICAL SPACES WITH NO COMPACTLY DETERMINED EXTENSIONS

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The $\mathcal{P}$-spaces with no compactly determined $\mathcal{P}$-extensions are characterized for $\mathcal{P} = LM_2$, $\mathcal{T}_2$ and $\mathcal{T}_3$. It is proved that every $LM_2$-closed space is compact. Examples of absolutely $LM_2$-closed spaces which are not compact are given.

An extension $\nu$ of a topological space $X$ is called a compactly determined extension if for every point $y \in \nu \setminus X$ there is $x \in X$ such that $y \in \nu_x$ and $\nu_x$ is compact [3]. All compactly determined $\mathcal{P}$-extensions of a $\mathcal{P}$-space for $\mathcal{P} = \mathcal{T}_2$ and $\mathcal{P} = \mathcal{T}_2$ were characterized by L. Dostchinov [11] and I. Gotchev [12] in terms of supertopologies. In this paper the $\mathcal{P}$-spaces with no compactly determined $\mathcal{P}$-extensions are characterized for $\mathcal{P} = LM_2$, $\mathcal{T}_2$ and $\mathcal{T}_3$. A space $X$ is an $LM_2$-space if every compact subspace of $X$ is $\mathcal{T}_2$ [7]. Obviously $\mathcal{T}_2 \subseteq LM_2 \subseteq \mathcal{T}_4$. For $\mathcal{P} = LM_2$ the spaces with no compactly determined $LM_2$-extensions coincide with the absolutely $LM_2$-closed spaces. This fact was observed by B. Nikolov and E. Gulić. A $\mathcal{P}$-space $X$ is absolutely $\mathcal{P}$-closed if for every $\mathcal{P}$-space $Z \in \mathcal{P}$ and for every embedding $h : X \to Z$ there exist a space $Y \in \mathcal{P}$ and two continuous maps $f, g : Z \to Y$ such that $h(x) = \{z \in Z | f(z) = g(z)\}$ (see [1], [12] and [13]). A $\mathcal{P}$-space $X$ is $\mathcal{P}$-closed if $X$ is a closed subspace in every $\mathcal{P}$-space in which it is embedded. In this paper also we will show that every $LM_2$-closed space is compact. Examples of absolutely $LM_2$-closed spaces which are not compact are given, which answer a question of B. Dikranjan and E. Gulić.

In this paper every topological space is $\mathcal{T}_2$ and every proximity $\delta$ in a topological space $X$ will be compatible with the topology on $X$. (For the definition of proximity, see [14].) Let us recall that in a space $X$ there exists a proximity $\delta$ if and only if $X$ is a completely regular space.

Let $X$ be a topological space, a subset $A$ of $X$ is $\delta$-closed [8] if for every point $x \in X \setminus A$ there is a neighborhood $U$ of $x$ such that $U \cap A = \emptyset$ and $A$ is $\delta$-closed if for every closed set $F \subseteq X \setminus A$ there is an open set $U$ of $X$ such that $FU = \emptyset$ and $\overline{U} \cap A = \emptyset$.

Obviously, every $\delta$-closed subset of $X$ is $\delta$-closed and every $\delta$-closed subset of $X$ is closed. If $X$ is regular then every closed set in $X$ is $\delta$-closed and if $X$ is normal then every closed set in $X$ is $\delta$-closed.

Let $A \subseteq X$ and $\delta$ be a proximity in $A$, we say that $A$ is $\delta$-placed in $X$ if for every pair $B, C \subseteq A$ such that $B \cap C$ there are disjoint open sets $U$ and $V$ in $X$ with $BU \subseteq A$ and $CV \subseteq A$.

It is clear that if $A$ is $\delta$-placed in $X$ then $A$ is a completely regular space and if $X$ is a Hausdorff space then every compact subset $B$ of $X$ is
A subset \( A \) of \( X \) is compactly related to \( X \) if every filter of closed sets on \( X \) has empty intersection intersect \( A \).

Let \( X \) and \( Y \) be topological spaces, \( A \) be a closed set in \( X \) and \( f : A \to Y \) be a mapping. Then the adjunction space \( X \cup_f Y \) (see [4]) is denoted by \( X \cup_f A \) in the case \( A \subseteq Y \) and \( f : A \to Y \) is the identical embedding.

**Lemma 1.** Let \( A \) be a closed set in \( X \) and \( \delta A \) be a \( T_1 \)-compactification of \( A \).

Then \( X \) is compactly related to \( A \) if and only if \( X \cup_f \delta A \) is compact.

Proof: Let \( X \) be compactly related to \( A \) and let \( Y = X \cup_f \delta A \). We prove that \( Y \) is compact.

Let \( \mathcal{F} \) be an ultrafilter of closed sets on \( Y \). If there exists \( F \in \mathcal{F} \) such that \( F \cap X = \emptyset \), then \( F \cap \delta A = \emptyset \). Moreover \( A \) is a closed set in \( X \), hence \( \delta A \) is a closed set in \( Y \). Since \( \delta A \) is compact and \( F \cap \delta A = \emptyset \), then \( \mathcal{F} \) has a cluster point in \( \delta A \) and hence \( \mathcal{F} \) has a cluster point in \( Y \). Let now for every \( F \in \mathcal{F} \), \( F \cap X = \emptyset \). Then \( \mathcal{F}' = \{ F \cap X \mid F \in \mathcal{F} \} \) is an ultrafilter of closed sets in \( X \). If there exists \( F \in \mathcal{F}' \) such that \( F \cap X = \emptyset \), then \( F \cap \delta A = \emptyset \) and hence \( \mathcal{F} \) has cluster points in \( \delta A \) and hence \( \mathcal{F} \) has a cluster point in \( Y \). If every \( F \in \mathcal{F} \) meets \( \forall x \in Y \) then \( \mathcal{F} = \{ F : F \cap X = \emptyset \} \) is an ultrafilter of closed sets in \( X \). Suppose there exists \( F \in \mathcal{F} \) such that \( F \cap X = \emptyset \). Since \( X \) is a compactly related to \( A \) and consequently also \( \mathcal{F} \) has cluster points in \( Y \). If every \( F \in \mathcal{F} \) meets \( \forall x \in Y \setminus X \) then \( \mathcal{F} = \{ F : F \cap X = \emptyset \} \) is an ultrafilter of closed sets in \( X \). Then \( \mathcal{F} = \{ F : F \cap X = \emptyset \} \) and hence \( \mathcal{F} \) has a cluster point in \( Y \). Therefore \( Y \) is compact.

Now let \( Y = X \cup_f \delta A \) be compact. We will prove that \( X \) is compactly related to \( A \).

Let \( \mathcal{F} \) be an ultrafilter of closed sets in \( X \) and assume that \( \mathcal{F} \) has no cluster points. The space \( X \) is compact, hence the filter \( \mathcal{F}' = \{ F : F \cap X = \emptyset \} \) has a cluster point \( \infty \) in \( Y \). Since \( \infty \notin X \setminus X \), \( \mathcal{F} \) has a cluster point \( \infty \) in \( Y \).

Let us assume that there exists a closed set \( F \in \mathcal{F} \) such that \( F \cap A = \emptyset \). Then \( \infty \in X \setminus X \) is an open set in \( X \) and \( F \cap X = \emptyset \). Thus \( \infty \in Y \setminus X \) is an open neighborhood of \( \infty \) in \( Y \) and \( F \cap X = \emptyset \). Therefore \( \infty \in Y \setminus X \) and \( \infty \notin X \). Hence \( \infty \notin X \).

**Corollary.** Let \( A \) be a compact subspace of \( X \). Then \( X \) is compactly related to \( A \), if and only if \( X \) is compact.

**Lemma 2.** Let \( X \) be a Hausdorff space, \( A \) be a closed set of \( X \), \( \delta \) be a proximity in \( A \) and \( \delta A \) be the compactification of \( A \) related to \( \delta \) (see [4]). Then \( X \cup_f \delta A \) is a Hausdorff space if and only if \( A \) is \( \theta \)-closed and \( \delta \)-placed in \( X \).

Proof: Let \( Y = X \cup_f \delta A \) be a Hausdorff space. We shall prove that \( A \) is \( \delta \)-closed in \( X \). Let \( B \subseteq A \) and \( B \delta C \). It is easily seen that \( B \delta C \) if and only if \( \delta C \) is Hausdorff and compact, so there exist open sets \( U \) and \( V \) in \( Y \) such that \( B \subseteq U \), \( \delta C \subseteq V \) and \( U \cap V = \emptyset \). So if \( U \cap V = \emptyset \) then \( \infty \in Y \setminus X \), \( \delta A \subseteq \emptyset \setminus \delta A \) and \( \delta A \subseteq \emptyset \setminus \delta A \) and \( \infty \notin X \). Hence \( Y \) is a Hausdorff space, there exist open sets \( U \) and \( V \) in \( Y \) such that \( x \in U \), \( \delta A \subseteq V \) and \( U \cap V = \emptyset \). Thus \( Y \) is a Hausdorff space if and only if \( A \) is \( \theta \)-closed and \( \delta \)-placed in \( X \).

Now let \( A \) be \( \theta \)-closed and \( \delta \)-placed in \( X \). We will prove that \( Y = X \cup_f \delta A \) is a Hausdorff space. Let \( x \in U \), \( \delta A \subseteq V \) and \( \delta A \subseteq \emptyset \setminus \delta A \). The space \( X \setminus A \) is
open in $Y$ and $X$ is a Hausdorff space, so there exist disjoint open sets $U$ and $V$ in $Y$ such that $x_1 \in U$ and $x_2 \in V$.

Now let $x_2 \in \delta A$. Since $A$ is a $\delta$-closed set in $X$ then there exist open sets $U$ and $V$ in $X$ such that $x_2 \in U$, $A \subset V$, and $U \cap \overline{V} = \emptyset$. Hence if $W = UV \cap \overline{V} \cap \emptyset$ then $W$ is an open neighbourhood of $x_2$ in $X$. $U$ is an open neighbourhood of $x_2$ in $Y$ and $U \cap \overline{V} \cap \emptyset$. Let now $x_2 \in \delta A$. Since $\delta A$ is a closed Hausdorff subspace in $Y$, there exist open sets $U_1$ and $U_2$ in $\delta A$ such that $U_1 \cap \overline{U_2} = \emptyset$. Let $F_1 = \overline{U_1} \cap \overline{U_2} \cap \emptyset$ and $F_2 = \overline{U_1} \cap \overline{U_2} \cap \emptyset$. Clearly $F_1 \cap F_2$. Since $A$ is $\delta$-closed in $X$ then there exist open sets $U'_1$ and $U'_2$ in $X$ such that $F_1 \cap U'_1 \cap \overline{U'_2} = \emptyset$ and $U'_1 \cap \overline{U'_2} = \emptyset$. On the other hand there exist open sets $U''_1$ and $U''_2$ in $X$ such that $U''_1 \cap \overline{U''_2} = \emptyset$. Thus if $V_1 = U''_1 \cap \overline{U''_2} = \emptyset$ and $V_2 = U''_1 \cap \overline{U''_2} = \emptyset$ then $V_1$ and $V_2$ are disjoint open sets in $Y$ such that $x_1 \in V_1$ and $x_2 \in V_2$. Therefore $Y$ is a Hausdorff space.

The following lemma could be proved in the same way as the above lemma.

**Lemma 1.** Let $X$ be a regular space, $A$ be a closed set in $X$, $\delta$ be a proximity in $A$ and $\overline{\delta} A$ be the compactification of $A$, related to $\delta$.

Then $\delta A \overline{\delta} A$ is a regular space if and only if $A$ is $\delta$-closed and $\delta$ placed in $X$.

**Theorem 1.** Let $X$ be an $LM_2$-space. Then $X$ has no compactly determined $LM_2$-extension if and only if every subset $A$ of $X$ which satisfies the following two conditions is compact.

i) Every compactly related to $A$ subset of $X$ is a Hausdorff space.

ii) There exists a proximity $\delta$ in $A$ such that $A$ is $\delta$-closed and $\delta$ placed in every compactly related to $A$ subset of $X$.

Proof. Let us assume that there exists a compactly determined $LM_2$-extension $Y$ of $X$ and $X \neq X$. We may assume without loss of generality that there exists a closed set $A$ in $X$ such that $\overline{\delta} Y$ is compact and $Y = XU\overline{\delta} Y$. We shall prove that $A$ satisfies the conditions i) and ii). Let $Z \subset X$ be compactly related to $A$. Then $A \subset Z$ and $\delta$ will be a closed set in $Z$. Furthermore $ZU\overline{\delta} Y$ will be homeomorphic to $ZU\overline{\delta} Y$. By Lemma 1 we know that $ZU\overline{\delta} Y$ is compact. Thus $ZU\overline{\delta} Y$ is a compact subspace of $Y$. Since $Y$ is $LM_2$, then $ZU\overline{\delta} Y$ is a Hausdorff space and hence $Z$ is a Hausdorff space. If $\delta$ is the proximity of $A$ induced by the standard proximity $\delta'$ of the compact space $\overline{\delta} Y$ then $A$ will be $\delta'$-closed and $\delta'$ placed in $Z$ by Lemma 2. Therefore $A$ satisfies the conditions i) and ii) but $A$ is not compact - a contradiction.

Now assume that there exists $A \subset X$ such that the conditions i) and ii) are satisfied and $A$ is not a compact space. Since $A$ is compactly related to $A$ ii) yields that $A$ is a compactly regular space. Furthermore if $A \subset X$ then $A \cup \{x\}$ is compactly related to $A$ by i) $A$ is $\delta$-closed in $A \cup \{x\}$. Hence $A$ is a closed set in $X$. Let $\overline{\delta}$ be the proximity in $A$ which satisfies ii) and let $\overline{\delta} A$ be the compactification of $A$ related to $\overline{\delta}$. Set $Y = XU\overline{\delta} A$, clearly $Y$ is a compactly determined extension of $X$. We will prove that $Y$ is an $LM_2$-space. Let $Z \subset Y$ be a compact subspace. Without loss of generality we may assume that $\overline{\delta} A \subset Z$. For $Z \subset Z\overline{\delta} A$ the space $ZU\overline{\delta} A$ is homeomorphic to $Z\overline{\delta} A$. By Lemma 1 and by the compactness of $ZU\overline{\delta} A$ it follows that $Z$ is compactly related to $A$. But by i) $Z$ is a Hausdorff space and by Lemma 2 $Z$ is a Hausdorff space.

The above result is in fact a characterization of the absolutely $LM_2$-closed spaces.

**Theorem 2.** A Hausdorff space $X$ has no Hausdorff compactly determined extensions if and only if every $\delta$-closed and $\delta$-placed subset of $X$ is compact.
Proof. Follows by Lemma 2.

Theorem 3. A regular space $X$ has no regular compactly determined extensions if and only if every $G$-closed and $G$-placed subset of $X$ is compact.

Proof. Follows by Lemma 3.

Now we will show that every $LM_2$-closed space is compact.

Lemma 4. Let $\infty$ be a non-isolated point for a topological space $X$. If $\infty$ has no compact neighborhood in $X$, then $X$ is not an $LM_2$-closed space.

Proof. Let us assume that for some non-isolated point $\infty \in X$ there is no open set $U$ in $X$ such that $\infty \in U$ and $\overline{U}$ is compact. We consider the space $Y = X \cup \{y\}$ where $y \notin X$ with the following topology: $\{y\}$ is a closed set and for a base of neighborhoods of $\{y\}$ we take the family \( \{U \cup (y \cap F) \cup \{y\}\} \), where $V$ is an open set in $X$, $\infty \in V$ and $F$ is a compact set in $X$. It is easily seen that it is a topology on $Y$. We will prove that $Y$ is an $LM_2$ space. Let $A \subseteq Y$ and $A$ be a compact subspace. If $y \notin A$ then $A \subseteq X$ and $X$ is Hausdorff compact. Now let $y \in A$, then $\{y\} \cup A$ is compact. If $U_X$ and $U_y$ are the filters of neighborhoods of $\infty$ and $y$ on $Y$ then for every $U \in U_X$ we have $U \cup \{y\} \in U_Y$. Thus $F = (\overline{U} \cup \{y\}) \setminus \{y\}$ is a compact space in $X$ and hence $F$ is a closed set in $X$. Thus there exists a neighborhood $U$ of $\infty$ in $X$ such that $U \setminus \{y\} \neq \emptyset$. Therefore $W = \{y\} \cup (U \setminus F)$ is a neighborhood of $y$ in $Y$ avoiding $F$, hence $y$ is an isolated point in $A$. Thus $Y$ is an $LM_2$-extension of $X$, then $X$ is not an $LM_2$-closed space.

Lemma 5. Let $X$ be an $LM_2$-space. If every point in $X$ has a compact neighborhood in $X$, then $X$ is a Hausdorff space.

Proof. It is obvious.

Theorem 4. An $LM_2$-space $X$ is $LM_2$-closed if and only if $X$ is a compact Hausdorff space.

Proof. It is obvious that if $X$ is a compact Hausdorff space then $X$ is an $LM_2$-closed space. Now let $X$ be an $LM_2$-closed space. By Lemma 4 it follows that every point $x \in X$ has an open neighborhood with compact closure. Thus by Lemma 5 $X$ is a locally compact Hausdorff space. Therefore $X$ is a completely regular space and by the closedness, $X$ is a compact.

The following example shows that there exists an absolutely $LM_2$-closed space which is not compact.

Example. Let $X = [0, 1]$ and $\mathcal{T}$ be the usual topology on $X$. Let $\mathcal{T}_f$ be the coarsest topology on $X$ such that $\mathcal{T} \subseteq \mathcal{T}_f$ and $\{\frac{1}{n}\}_{n=1}^{\infty}$ is a closed set in $\mathcal{T}_f$. Then $(X, \mathcal{T}_f)$ is $LM_2$-closed but it is not even countably compact.

Proof. The absolute $LM_2$-closedness of $(X, \mathcal{T}_f)$ may be proved by means of Theorem 1. Obviously $X$ is not countably compact.

Other examples of $LM_2$ spaces which have no compactly determined extensions and which are not Hausdorff compact may be obtained from the following result which we give without proof.

Theorem 5. Let $X$ be a first countable compact Hausdorff space and there are two dense, disjoint sets $D_1$ and $D_2$ such that $D_1 \cup D_2 = X$. Let $\mathcal{T}_f$ be the coarsest topology on $X$ such that $\mathcal{T} \subseteq \mathcal{T}_f$ and $D_1 \in \mathcal{T}_f$. Then $(X, \mathcal{T}_f)$ is absolutely $LM_2$-closed.

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