CONTINUOUS EXTENSIONS OF FUNCTIONS DEFINED ON SUBSETS OF PRODUCTS WITH THE κ-BOX TOPOLOGY

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ABSTRACT. Consider these results: (a) [N. Noble, 1972] *every G_δ-dense subspace in a product of separable metric spaces is C-embedded*; (b) [M. Ulmer, 1970/73] *every Σ-product in a product of first-countable spaces is C-embedded*; (c) [R. Pol and E. Pol, 1976, also A. V. Arhangel′skiĭ, 2000, as corollaries of more general theorems] *every dense subset of a product of completely regular, first-countable spaces is C-embedded in its G_δ-closure.*

The present paper continues the authors' earlier initiative [Continuous extensions of functions defined on subsets of products, *Topology and Its Applications*, **159** (2012), 2331–2337], which already generalized those cited results in several ways simultaneously (*e.g.*, κ-box topology on the product spaces; relaxed separation properties on both the domain and the range spaces). Now the authors show:

Let $\kappa \leq \alpha$ satisfy $\lambda < \kappa$, $\beta \leq \alpha \Rightarrow \beta^\lambda \leq \alpha$; let Y be dense in an open subset U of a κ-box product $(\Pi_{i\in I} X_i)_\kappa$ with each X_i a T_1-space; let $q \in X_I \backslash Y$ have the property that for each $J \in [I]^{\leq\alpha}$ there is $y \in Y$ such that $y_J = q_J$; let Z be a regular space with a $\overline{G_{\alpha^+}}$-diagonal. Suppose that for each $i \in I$ either $\chi(q_i, X_i) \leq \alpha$ or each intersection of κ-many neighborhoods of q_i is another such neighborhood. Then every continuous $f : Y \to Z$ extends continuously over $Y \cup \{q\}$.

Several corollaries and consequences are given.

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1. Introduction

Topological spaces considered here are not subjected to any standing separation properties. Additional hypotheses are imposed as required. Throughout this paper, ω is the least infinite cardinal, κ and α are infinite cardinals. For I a set we define $[I]^{<\kappa} := \{J \subseteq I : |J| < \kappa\}$, the symbol $[I]^{\leq\kappa}$ is defined analogously. For X a space and $x \in X$, a set $U \subseteq X$ is a *neighborhood* of x in X if x is in the interior of U in X. For $A \subseteq X$ we denote by $\mathcal{N}_X(A)$, or simply by $\mathcal{N}(A)$ when ambiguity is unlikely, the set of open sets in X containing A. A point $x \in X$ is a *$P(\kappa)$-point* of X if $\cap\mathcal{V} \in \mathcal{N}(x)$ whenever $\mathcal{V} \subseteq \mathcal{N}(x)$ and $|\mathcal{V}| < \kappa$; X is a *$P(\kappa)$-space* provided each point $x \in X$ is a $P(\kappa)$-point. Clearly, every topological space is a $P(\omega)$-space. The $P(\omega^+)$-spaces are often called *P-spaces* (cf. [10] and sources cited there).

For X a space, $x \in X$ and $A \subseteq X$, x belongs to the *G_κ-closure* of A in X if $(\cap\mathcal{V}) \cap A \neq \emptyset$ whenever $\mathcal{V} \subseteq \mathcal{N}(x)$ and $|\mathcal{V}| < \kappa$. A set $A \subseteq X$ is a *G_κ-set* [resp., a *$\overline{G_\kappa}$-set*] in X if there exists $\mathcal{V} \subseteq \mathcal{N}_X(A)$ such that $|\mathcal{V}| < \kappa$ and $A = \cap\mathcal{V}$ [resp., $A = \cap\{\overline{V} : V \in \mathcal{V}\}$]. (Thus, the familiar G_δ-sets are exactly the G_{ω^+}-sets.) V is a *G_κ-neighborhood of* x if there exists a G_κ-set U such that $x \in U \subseteq V$.

For spaces X and Z and $Y \subseteq X$, the symbol $C(Y,Z)$ denotes the set of all continuous functions $f : Y \to Z$. The subspace Y of X is *$C(Z)$-embedded* in X provided each function $f \in C(Y,Z)$ extends continuously over X; when $Z = \mathbb{R}$, Y is said to be *C-embedded* in X; if Y is $C(Z)$-embedded in Y for every metrizable space Z then Y is said to be **M**-*embedded* in X.

Below we use the simple fact (which we will not mention again explicitly) that when Y is dense in X and Z is a regular T_1-space, a function $f \in C(Y,Z)$ extends continuously over X if and only if f extends continuously to each point of $X\backslash Y$ (restated: Y is $C(Z)$-embedded in X if and only if Y is $C(Z)$-embedded in $Y \cup \{q\}$ for each $q \in X\backslash Y$); in this connection see [1], [12].

For a set $\{X_i : i \in I\}$ of sets and $J \subseteq I$, we write $X_J := \Pi_{i\in J} X_i$; and for every *generalized rectangle* $A = \Pi_{i\in I} A_i \subseteq X_I$ the *restriction set of* A, denoted $R(A)$, is the set $R(A) = \{i \in I : A_i \neq X_i\}$. When each $X_i = (X_i, \mathcal{T}_i)$ is a space, the symbol $(X_I)_\kappa$ denotes X_I with the *κ-box topology*; this is the topology for which $\{U : U = \Pi_{i\in I} U_i, U_i \in \mathcal{T}_i, |R(U)| < \kappa\}$ is a base. Thus the ω-box topology on X_I is the usual product topology. We note that even when κ is regular, the intersection of fewer than κ-many sets, each open in $(X_I)_\kappa$, may fail to be open in $(X_I)_\kappa$.

A (not necessarily faithfully) indexed family $\{A_i : i \in I\}$ of nonempty subsets of a space X is *locally* $< \kappa$ if there is an open cover $\mathcal{U}$ of X such that $|\{i \in I : U \cap A_i \neq \emptyset\}| < \kappa$ for each $U \in \mathcal{U}$. A space $X = (X, \mathcal{T})$ is *pseudo*-(α, κ)-*compact* if every indexed locally $< \kappa$ family $\{U_i : i \in I\} \subseteq \mathcal{T} \backslash \{\emptyset\}$ satisfies $|I| < \alpha$, and X is *pseudo*-α-*compact* if it is *pseudo*-(α, ω)-*compact*. In this terminology, the familiar pseudocompact spaces are the pseudo-ω-compact spaces.

For $p \in X_I = \Pi_{i \in I} X_i$ and $\alpha \geq \omega$ the α-Σ-*product of* X_I *based at* p is the set $\Sigma_\alpha(p) := \{x \in X_I : |\{i \in I : x_i \neq p_i\}| < \alpha\}$. Usually $\Sigma_\omega(p)$ is denoted by $\sigma(p)$.

The notation $\kappa \ll \alpha$ means that α *is strongly* κ-*inaccessible.* That is: (a) $\kappa < \alpha$, and (b) $\beta^\lambda < \alpha$ whenever $\beta < \alpha$ and $\lambda < \kappa$ (see [7, p. 254]).

For (standard) topological definitions and notation not repeated here, see [8], [10], or [7].

Remark 1.1. The M.A. thesis [13], written by the third-listed co-author under the guidance of the second-listed co-author, incorporates many of the findings of this paper. The reader might consult that for commentary supplementing the historical background given in Section 2 below.

2. Historical background. Ulmer's theorem

The point of departure for our work is the following theorem of Ulmer [15], [14]. (We state his theorem using the notation introduced here.)

Theorem 2.1. [15, Theorem 2.2] *Let* $\{X_i : i \in I\}$ *be a set of Tychonoff spaces with* $p \in X_I$ *and* $\alpha \geq \omega$ *be a cardinal. Suppose that either*

(i) $\Sigma_\alpha(p)$ *is pseudo-*α*-compact; or*
(ii) α *is uncountable and* $x \in X_i$ *implies* $\chi(x, X_i) < \alpha$*; or*
(iii) α *is uncountable and every* X_i *is a* P*-space.*

Then $\Sigma_\alpha(p)$ *is* C*-embedded in* X_I*.*

We note here that if $\alpha = \omega$ in Theorem 2.1(i), so that $\Sigma_\alpha(p) = \sigma(p)$, then $\sigma(p)$ must be a pseudocompact dense subset of X_I. But that cannot happen if $|I| \geq \omega$ since $\sigma(p)$ will not be a G_δ-dense subset of X_I.

Theorem 2.1 has been generalized in several different ways over the years by many authors. Here we shall mention mainly those results that generalize Theorem 2.1 for the usual product topology and for the κ-box topology. For results not mentioned here we refer the reader to [11], [7] and to the historical remarks contained in the other papers mentioned below.

(a) W. W. Comfort and S. Negrepontis in [6], and later in [7, 10.4], generalized the case of regular $\alpha > \omega$ of (i) for the case of κ-box topology assuming that $(X_I)_\kappa$ is a pseudo-(α, κ)-compact and $\Sigma_\alpha(p)$ is replaced by $Y \subseteq X_I$ such that $\pi_J[Y] = X_J$ for all nonempty $J \in [I]^{<\alpha}$; and C-embedded is replaced by $\mathbf{M}$-embedded.

It must be noted here that the proofs in [7] depend on [7, 10.1], for which the proof was incomplete. In [5] the authors gave a complete proof of [7, 10.1] where no separation axioms were assumed for X_I. That allowed the authors to verify and unify all the results from [7, Chapter 10] whose status had become questionable, and to extend several of these. Later in [2] the results from [7] were generalized even further by replacing $\mathbf{M}$-embedded with $C(Z)$-embedded, where Z is a space such that the diagonal in Z is the intersection of $< \alpha$-many regular-closed subsets of $Z \times Z$. Those results generalized to the κ-box topology also some results from [11] obtained there for the usual product topology (for more specific information see [2]).

(b) The case $\alpha = \omega^+$ of Theorem 2.1(ii) has been generalized in many different ways by many authors (see [3] for more information). In [4], Theorem 2.1(ii) was generalized for the case of κ-box topology and $\alpha = \kappa^+$, assuming that each $x \in X_I$ is a $P(\kappa)$-point. In [3] the result from [4] and the case $\alpha = \omega^+$ of Theorem 2.1(ii) were generalized in the following way. (For the definition when a subset Y of X_I α^+-duplicates a point $q \in X_I$, see 3.2 below.)

Theorem 2.2. [3, 3.10] *Let $\omega \leq \kappa \leq \alpha$ with either $\kappa < \alpha$ or α regular, let $\{X_i : i \in I\}$ be a set of T_1-spaces, and let Y be dense in an open subset of $(X_I)_\kappa$. Let $q \in X_I \setminus Y$ be such that (i) Y α^+-duplicates q in X_I and (ii) q_i is a $P(\alpha)$-point in X_i with $\chi(q_i, X_i) \leq \alpha$ for each $i \in I$. Then Y is $C(Z)$-embedded in $Y \cup \{q\}$ for each regular space Z with a $\overline{G_{\alpha^+}}$-diagonal.*

(c) Theorem 2.2 does not generalize Theorem 2.1(iii); indeed, we are not aware of any published generalizations of Theorem 2.1(iii). Addressing that challenge, now in Theorem 3.10 and its corollaries (Theorems 3.11 and 3.12) we generalize Theorem 2.1(iii) in several directions simultaneously. We show further that in Theorem 2.2 the conjunction of the two hypotheses on the points $q_i \in X_i$, namely that each is a $P(\alpha)$-point and that each satisfies $\chi(q_i, X_i) \leq \alpha$, can be greatly relaxed: to achieve the desired conclusion it is enough for each $i \in I$ that q_i is a $P(\kappa^+)$-point of X_i or that $\chi(q_i, X_i) \leq \alpha$.

3. Main Results

Definition 3.1. Let $\kappa \geq \omega$ and let Z be a space. Then

(a) $\Delta(Z) := \{(z, z) \in Z \times Z : z \in Z\}$ is the *diagonal* of Z; and

(b) Z has a G_κ-*diagonal* [resp., a $\overline{G_\kappa}$-*diagonal*] if $\Delta(Z)$ is a G_κ-set [resp., a $\overline{G_\kappa}$-set] in $Z \times Z$.

The terminology given in 3.2, also Theorem 3.4, appeared in [3, §3.5–3.8].

Notation and Definition 3.2. Let $\alpha \geq \omega$, and let $\{X_i : i \in I\}$ be a family of sets, $Y \subseteq X_I := \Pi_{i\in I} X_i$ and $q \in X_I$. Then

(a) For $\emptyset \neq J \subseteq I$ we set $Y_{q_J} := \{y \in Y : q_J = y_J\}$; and

(b) Y α-*duplicates* q if $\emptyset \neq J \in [I]^{<\alpha} \Rightarrow Y_{q_J} \neq \emptyset$.

Remarks 3.3. (a) The knowledgeable reader will have speculated (correctly) that the authors of [3] introduced the concept of α-duplication as a useful generalization of the familiar Σ_α-products and subspaces $Y \subseteq X_I$ such that $\pi_J[Y] = X_J$ for all nonempty $J \in [I]^{<\alpha}$.

(b) To help fix ideas further, we note that if $Y \subseteq X_I$ with each X_i a discrete space, then $Y \subseteq X_I$ is dense in $(X_I)_\alpha$ if and only if Y α-duplicates each $q \in X_I$. More generally we have, as remarked in [3]: If $\kappa \leq \alpha^+$ and q is a point in X_I such that $\psi(q_i, X_i) \leq \alpha$ for each $i \in I$, then Y α^+-duplicates q in X_I if and only if q belongs to the G_{α^+}-closure of Y in $(X_I)_\kappa$.

(c) If in (b) the condition $\psi(q_i, X_i) \leq \alpha$ is replaced by the condition that each q_i is a $P(\kappa^+)$-point of X_i (see Theorem 3.10(i) below) then, even when $\alpha = \kappa = \omega$, a G_{α^+}-dense subset Y of a space $(X_I)_\kappa$ need not α^+-duplicate every point of $(X_I)_\kappa$. For an example, it is enough to take $X = D \cup \{q\}$ with D discrete, $|D| = \omega^+$, neighborhoods of q are co-countable, and $Y := D^{\omega^+} \subseteq X^{\omega^+} = (X^{\omega^+})_\omega$.

For what follows we need the following results.

Theorem 3.4. [3] *Let $\omega \leq \kappa \leq \alpha$ with either $\kappa < \alpha$ or α regular, let $\{X_i : i \in I\}$ be a set of spaces, and let Y be dense in an open subset U of $(X_I)_\kappa$. Let $q \in X_I \setminus Y$ be a point such that Y α^+-duplicates q, and let Z be a space with a $\overline{G_{\alpha^+}}$-diagonal. Then for every $f \in C(Y, Z)$ there are $S \in [I]^{\leq\alpha}$ and $z \in Z$ such that $f(y) = z$ whenever $y \in Y$ satisfies $y_S = q_S$.*

Definition 3.5. Let X, Y and Z be spaces, $Y \subseteq X$, $V \subseteq Z$, and $f : Y \to Z$. Then
(a) $W \subset X$ is V-*small* (for f) if $f[W \cap Y] \subseteq V$; and
(b) if $y \in Y$, $W \in \mathcal{N}_X(y)$, and W is V-small, then W is a V-*neighborhood* of y.

Definition 3.6. Let $\omega \leq \kappa \leq \alpha$, let $\{X_i : i \in I\}$ be a set of spaces, and let Y be a subset of $(X_I)_\kappa$. Let $q \in X_I$, V be open in a space Z, and $f \in C(Y, Z)$. Then $J \subseteq I$ is V_α-*cofinal for* q if (i) $|J| < \alpha$ and (ii) for every $J' \in [I \backslash J]^{<\alpha}$ there exist $y \in Y_{q_J}$ and a basic open V-neighborhood W of y in $(X_I)_\kappa$ such that $R(W) \cap J' = \emptyset$.

We acknowledge in greater detail our substantial structural and intellectual debt to Ulmer [15], whose Theorem 2.2 is generalized below in our Theorem 3.10 and its corollaries, Theorems 3.11 and 3.12. First, our Definition 3.6 (of a V_α-cofinal set $J \subseteq I$) closely parallels, and is motivated by, his Definition 2.5 of an ϵ_γ-cofinal set $J \subseteq I$ when $\alpha = \aleph_\gamma$; our Lemma 3.7, showing the existence for $V \in \mathcal{N}_Z(z)$ of a special V_{α^+}-cofinal set $J_V \in [I]^{\leq\alpha}$ for q, parallels his Lemma 2.6; and our surprising Lemma 3.8, showing that there is a V_{α^+}-cofinal set $K_V \subseteq I$ for q of cardinality $< \kappa$, precisely parallels his construction (in his more limited context) of a finite ϵ_γ-cofinal set $J \subseteq I$.

Though our new concepts and results, then, clearly parallel Ulmer's, there are substantial differences. Strictly considered our concept of a V_{α^+}-cofinal set does not formally generalize Ulmer's concept of an ϵ_γ-cofinal set since in our case we define the concept for every point $q \in X_I$ (and we use this concept only for points $q \notin Y$), while Ulmer (when $Y = \Sigma_\alpha(p) = \Sigma_{\aleph_\gamma}(p)$ and $Z = \mathbb{R}$)

restricts to the case $q \in Y$. When $q \notin Y$ then $f(q)$ is not defined and that is why in our definition we need a point y from Y_{q_J}. But then, even when our Definition 3.6 is altered to allow only the case $q \in Y = \Sigma_\alpha(p)$ and $Z = \mathbb{R}$, that definition and Ulmer's with $\alpha = \aleph_\gamma$ are still different because of the involvement of the set Y_{q_J}.

Lemma 3.7. *Let κ, α, X_I, Y, Z, U, q, f, S and z be as in Theorem 3.4. Then for every $V \in \mathcal{N}_Z(z)$ there exists $J_V \in [I]^{\leq\alpha}$ such that $S \subseteq J_V$ and J_V is V_{α^+}-cofinal for q.*

Proof. Suppose that for some $V \in \mathcal{N}_Z(z)$ no such J_V exists. Recursively for ordinals $\eta \leq \alpha$ we will define sets $S_\eta \in [I]^{\leq\alpha}$ with $S_\eta \supseteq S_\xi$ for all $\xi < \eta$.

Set $S_0 := S$ and suppose for $\eta < \alpha$ that $S_\xi \in [I]^{\leq\alpha}$ has been defined for all $\xi \leq \eta$, with $S_{\xi'} \subseteq S_\xi \subseteq S_\eta$ whenever $\xi' < \xi \leq \eta$. Since S_η is not V_{α^+}-cofinal for q, there is $J'_\eta \in [I \setminus S_\eta]^{\leq\alpha}$ such that for every $y \in Y_{q_{S_\eta}}$ and for every V-neighborhood W of y in $(X_I)_\kappa$ we have $R(W) \cap J'_\eta \neq \emptyset$. We set $S_{\eta+1} := S_\eta \cup J'_\eta$.

For limit ordinals $\eta \leq \alpha$ we set $S_\eta := \bigcup_{\xi<\eta} S_\xi$.

The recursive definitions are complete. We note that $|S_\alpha| \leq \alpha$.

Now, let $y \in Y_{q_{S_\alpha}}$ and let W be a basic open V-neighborhood of y in $(X_I)_\kappa$. Since $|R(W)| < \kappa$ and $\kappa < \alpha$ or α is regular, there exists $\eta_0 < \alpha$ such that $R(W) \cap (S_{\eta+1} \setminus S_\eta) = \emptyset$ for each η with $\eta_0 < \eta < \alpha$. But for each such η we have $y \in Y_{q_{S_\eta}}$ and hence $R(W) \cap (S_{\eta+1} \setminus S_\eta) \neq \emptyset$, a contradiction. □

Lemma 3.8. *Let $\omega \leq \kappa \ll \alpha^+$, let $\{X_i : i \in I\}$ be a set of spaces, and let Y be dense in an open subset U of $(X_I)_\kappa$. Let also $q \in X_I \setminus Y$ be a point such that Y α^+-duplicates q, and let Z be a space with a $\overline{G_{\alpha^+}}$-diagonal. Let $f \in C(Y, Z)$ and let S and z be as given by Theorem 3.4: $S \in [I]^{\leq\alpha}$, and $f(y) = z$ for all $y \in Y$ such that $y_S = q_S$. Then for every $V \in \mathcal{N}_Z(z)$ there exists $K_V \in [I]^{<\kappa}$ such that K_V is V_{α^+}-cofinal for q and $K_V \subseteq J_V$, where J_V is as given by Lemma 3.7.*

Proof. Given V, let J_V be as in Lemma 3.7: $J_V \in [I]^{\leq\alpha}$ and J_V is V_{α^+}-cofinal for q. If there exists $y \in Y_{q_{J_V}}$ and a basic V-neighborhood W of y in $(X_I)_\kappa$ with $R(W) \subseteq J_V$ then clearly $K_V := R(W)$ will be as required. Therefore, in what follows, we suppose that for every $y \in Y_{q_{J_V}}$ and every basic V-neighborhood W of y in $(X_I)_\kappa$ we have $R(W) \setminus J_V \neq \emptyset$.

Let $y_0 \in Y_{q_{J_V}}$ and let W_0 be a basic V-neighborhood of y_0 in $(X_I)_\kappa$. Then $R(W_0)\backslash J_V \neq \emptyset$. Since J_V is V_{α^+}-cofinal for q, there exists a point $y_1 \in Y_{q_{J_V}}$ and a basic V-neighborhood W_1 of y_1 in $(X_I)_\kappa$ with $R(W_1) \cap (R(W_0) \setminus J_V) = \emptyset$. According to our assumption, $R(W_1) \setminus J_V \neq \emptyset$ and therefore $R(W_1) \setminus (R(W_0) \cup J_V) \neq \emptyset$. We continue recursively: for $\eta < \alpha^+$ we choose $y_\eta \in Y_{q_{J_V}}$ and a basic open V-neighborhood W_η of y_η such that $R(W_\eta) \cap (\bigcup_{\xi<\eta} R(W_\xi) \setminus J_V) = \emptyset$. Then for every $\xi < \eta < \alpha^+$ we have $R(W_\xi) \cap R(W_\eta) \subseteq J_V$. Since $|R(W_\eta)| < \kappa$ and $\kappa \ll \alpha^+$, it follows from the Erdős-Rado theorem for quasi-disjoint families (see [9] or [7, 1.4]) that there are a set $A \subseteq \alpha^+$ with $|A| = \alpha^+$ and a set $K_V \in [I]^{<\kappa}$ such that $R(W_\xi) \cap R(W_\eta) = K_V \subseteq J_V$ for every $\xi, \eta \in A$, $\xi \neq \eta$.

To see that K_V is V_{α^+}-cofinal for q, let $K \in [I]^{\leq\alpha}$ satisfy $K_V \cap K = \emptyset$. Since $\{R(W_\eta) \setminus K_V : \eta \in A\}$ is a collection of α^+-many pairwise disjoint subsets of I, there exists an index $\eta_1 \in A$ such that $R(W_{\eta_1}) \cap K = \emptyset$. Then y_{η_1} and W_{η_1} are as required. □

Remark 3.9. We note that in Lemma 3.8 the possibility $K_V = \emptyset$ is not excluded. We claim in that case that if Z is regular then the function $\overline{f} : Y \cup \{q\} \to Z$ defined by

$$\overline{f}|Y = f, \ \ \overline{f}(q) = z$$

satisfies $\overline{f}[Y \cup \{q\}] \subseteq V$, so that every neighborhood of q in $Y \cup \{q\}$ is a V-neighborhood.

To verify that, let $V' \in \mathcal{N}_Z(z)$ be such that $\overline{V'} \subseteq V$ and let $\{R(W'_\eta) : \eta \in A\}$ be a family constructed as in the proof of Lemma 3.8 (now for the set V' in place of V): this is a family of α^+-many pairwise disjoint subsets of $I \setminus J_{V'}$, with each W'_η a basic open V'-neighborhood of some $y_\eta \in Y_{q_{J_{V'}}} \subseteq U$. Without loss of generality we can assume that $W'_\eta \subseteq U$ for each $\eta \in A$. Now suppose that there exists $y \in Y$ such that $f(y) \notin V$. Since f is continuous at y there exists a basic open neighborhood $W \subseteq U$ of y such that $f[W] \subseteq Z \setminus \overline{V'}$. Since $|R(W)| < \kappa$ there is $\eta \in A$ such that $R(W) \cap R(W'_\eta) = \emptyset$. Then $\emptyset \neq W \cap W'_\eta \subset U$, and since Y is dense in U there is $y_0 \in Y \cap W \cap W'_\eta$. Then from $y_0 \in W'_\eta$ we have $f(y_0) \in V'$, a contradiction since from $y_0 \in W$ we have also $f(y_0) \in Z \setminus \overline{V'}$.

Theorem 3.10. *Let $\omega \le \kappa \ll \alpha^+$, let $\{X_i : i \in I\}$ be a set of T_1-spaces, and let Y be dense in an open subset U of $(X_I)_\kappa$. Let $q \in X_I \setminus Y$ be a point such that Y α^+-duplicates q, and let Z be a regular space with a $\overline{G_{\alpha^+}}$-diagonal. Suppose for each $i \in I$ that either q_i is a $P(\kappa^+)$-point of X_i, or $\chi(q_i, X_i) \le \alpha$. Then Y is $C(Z)$-embedded in $Y \cup \{q\}$.*

Proof. Let $f \in C(Y, Z)$. Since Y α^+-duplicates q in X_I, we have from Theorem 3.4 that there exist $z \in Z$ and $J \in [I]^{\le\alpha}$ such that $f(y) = z$ for all $y \in Y$ satisfying $y_J = q_J$. We define $\overline{f} : Y \cup \{q\} \to Z$ by the rule

$$\overline{f}|Y = f,\ \overline{f}(q) = z.$$

We must show $\overline{f} \in C(Y \cup \{q\}, Z)$. Since Y is open in $Y \cup \{q\}$, the function $\overline{f}$ remains continuous at each $y \in Y$. To show that $\overline{f}$ is continuous at q we show that for each $V' \in \mathcal{N}_Z(z)$ there is a V'-neighborhood U' of q.

Since Z is regular, there is $V \in \mathcal{N}_Z(z)$ such that $\overline{V} \subseteq V'$. Let $J_V \in [I]^{\le\alpha}$ be as in Lemma 3.7: $S \subseteq J_V$ and J_V is V_{α^+}-cofinal for q. Let also $K_V \subseteq J_V$ be as in Lemma 3.8, *i.e.*, $K_V \in [I]^{<\kappa}$ and K_V is V_{α^+}-cofinal for q.

If $K_V = \emptyset$ then, according to Remark 3.9, the set $Y \cup \{q\}$ is a V'-neighborhood of q in $Y \cup \{q\}$, as required. If there exists a V-neighborhood W of some $y \in Y_{q_{J_V}}$ such that $R(W) \subseteq J_V$ then we are done. If not, then, as in the proof of Lemma 3.8, we choose recursively a transfinite sequence $\{y_\eta : \eta < \alpha^+\}$ of points such that $y_\eta \in Y_{q_{J_V}}$ and a transfinite sequence $\{W_\eta : \eta < \alpha^+\}$, where each W_η is a basic open V-neighborhood of y_η in $(X_I)_\kappa$ with the property that $R(W_\xi) \cap R(W_\eta) = K_V$ whenever $\xi < \eta < \kappa$.

For each $\eta < \alpha^+$, the set $\widetilde{W_\eta} := \Pi_{i\in K_V} (W_\eta)_i$ is a neighborhood of q_{K_V} in $(X_{K_V})_\kappa$.

We will define the required V'-neighborhood U' of q in $(X_I)_\kappa$ with the help of the following definition:

$$K_\chi := \{i \in K_V : \chi(q_i, X_i) \le \alpha\}.$$

If $K_\chi = \emptyset$ we set $B := \bigcap_{\eta<\kappa} \widetilde{W_\eta} = \Pi_{i\in K_V} (\bigcap_{\eta<\kappa} (W_\eta)_i)$. Since q_i is a $P(\kappa^+)$-point of X_i for each $i \in K_V$, we have that B is a neighborhood in $(X_{K_V})_\kappa$ of q_{K_V}.

If $K_\chi \neq \emptyset$, we claim that $\chi(q_{K_\chi}, (X_{K_\chi})_\kappa) \le \alpha$. Indeed for $i \in K_\chi$ let $\{(B_\eta)_i : \eta < \alpha\}$ be a base at q_i in X_i, and for $\phi \in \alpha^{K_\chi}$ let

$B(\phi) := \Pi_{i\in K_\chi}(B_{\phi(i)})_i$. Then $\mathcal{B} := \{B(\phi) : \phi \in \alpha^{K_\chi}\}$ is a base at q_{K_χ} in $(X_{K_\chi})_\kappa$, and $|\mathcal{B}| = \alpha$ since from $|K_\chi| < \kappa \ll \alpha^+$ we have $|\alpha^{K_\chi}| = \alpha$. Therefore, for each $\eta < \alpha^+$ there is $B(\phi) \in \mathcal{B}$ such that $B(\phi) \subseteq (\widetilde{W_\eta})_{K_\chi}$. It follows that there exist (fixed) $B_\chi \in \mathcal{B}$ and $\Lambda \in [\alpha^+]^{\alpha^+}$ such that $B_\chi \subseteq (\widetilde{W_\eta})_{K_\chi}$ for each $\eta \in \Lambda$. Now choose $\Lambda' \in [\Lambda]^\kappa$ and set $B := \bigcap_{\eta\in\Lambda'}\widetilde{W_\eta} = \Pi_{i\in K_V}(\bigcap_{\eta\in\Lambda'}(W_\eta)_i)$. Since q_i is a $P(\kappa^+)$-point of X_i for each $i \in K_V\backslash K_\chi$ and $B_\chi \subseteq (\widetilde{W_\eta})_{K_\chi}$ for each $\eta \in \Lambda'$, we have again that B is a neighborhood in $(X_{K_V})_\kappa$ of q_{K_V}.

The set B has been defined in each case. It remains to show that $U' := B \times \Pi_{i\in I\backslash K_V} X_i$ is a V'-neighborhood of q in $(X_I)_\kappa$.

If not, then there exists a point $y \in Y \cap U'$ such that $f(y) \notin V'$. Then $f(y) \in Z \setminus \overline{V}$ and since f is continuous at y we can find a basic $(Z \setminus \overline{V})$-neighborhood W' of y with $W' \subseteq U$. Note that when $K_\chi = \emptyset$ the family $\{R(W_\eta)\backslash K_V : \eta < \kappa\}$ is a family of κ-many pairwise disjoint subsets of I; and when $K_\chi \neq \emptyset$, the family $\{R(W_\eta) \setminus K_V : \eta \in \Lambda'\}$ is such a family. Then since $|R(W')| < \kappa$, there is η_0 (with $\eta_0 < \kappa$ when $K_\chi = \emptyset$, $\eta_0 \in \Lambda'$ otherwise) such that $(R(W_{\eta_0})\backslash K_V) \cap R(W') = \emptyset$ and hence $R(W_{\eta_0}) \cap R(W') \subseteq K_V$. Thus $W_{\eta_0} \cap W' \neq \emptyset$, and since Y is dense in U and $W' \subseteq U$ we have $Y \cap W_{\eta_0} \cap W' \neq \emptyset$. Then with $y' \in Y \cap W_{\eta_0} \cap W'$ we have this contradiction: $f(y') \in Z \setminus \overline{V}$ since $y' \in W'$, and $f(y') \in V$ since $y' \in W_{\eta_0}$. □

We note two consequences of Theorem 3.10 and Remarks 3.3(b).

Theorem 3.11. *Let $\omega \leq \kappa \ll \alpha^+$, let $\{X_i : i \in I\}$ be a set of T_1-spaces, and let Y be dense in an open subset U of $(X_I)_\kappa$. Let $q \in X_I \setminus Y$ be a point in the G_{α^+}-closure of Y in $(X_I)_\kappa$, and let Z be a regular space with a $\overline{G_{\alpha^+}}$-diagonal. If $\chi(q_i, X_i) \leq \alpha$ for each $i \in I$ then Y is $C(Z)$-embedded in $Y \cup \{q\}$.*

Theorem 3.12. *Let $\omega \leq \kappa \ll \alpha^+$, let $\{X_i : i \in I\}$ be a set of T_1-spaces, let Y be dense in an open subset U of $(X_I)_\kappa$, and let Z be a regular space with a $\overline{G_{\alpha^+}}$-diagonal. If every $q \in X_I \setminus Y$ in the G_{α^+}-closure of Y in $(X_I)_\kappa$ satisfies $\chi(q_i, X_i) \leq \alpha$ for each $i \in I$, then Y is $C(Z)$-embedded in its G_{α^+}-closure in $(X_I)_\kappa$.*

Remarks 3.13. (a) Suppose for a moment that in our principal result, Theorem 3.10, the hypothesis is (unnecessarily) strengthened to require that either

(i) q_i is a $P(\kappa^+)$-point of X_i, for each $i \in I$; or

(ii) $\chi(q_i, X_i) \leq \alpha$ for each $i \in I$.

Then when $\kappa = \omega$ and $Z = \mathbb{R}$ and Y is a Σ_{α^+}-product in X_I, the resulting two theorems slightly generalize Theorems 2.2.(iii) and 2.2.(ii) of [15], respectively. Similarly in this case, Theorem 3.12 slightly generalizes Theorem 2.2.(ii) of [15].

(b) In the same vein we note that Theorem 3.11, also the case when $\chi(q_i, X_i) \leq \alpha$ for each $i \in I$ of Theorem 3.10, compare with Theorem 3.10 of [3] as follows: the conclusions are the same, the hypothesis $\omega \leq \kappa \leq \alpha$ with either $\kappa < \alpha$ or α regular is strengthened to $\omega \leq \kappa \ll \alpha^+$, and the hypothesis that q_i is a $P(\alpha)$-point in X_i for each $i \in I$ is eliminated.

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