EMBEDDED PATHS AND CYCLES IN FAULTY HYPERCUBES

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ABSTRACT. An important task in the theory of hypercubes is to establish the maximum integer f_n such that for every set $\mathcal{F}$ of f vertices in the hypercube $\mathcal{Q}_n$, with $0 \le f \le f_n$, there exists a cycle of length at least $2^n - 2f$ in the complement of $\mathcal{F}$. Until recently, exact values of f_n were known only for $n \le 4$, and the best lower bound available for f_n with $n \ge 5$ was $2n-4$. We prove that $f_5 = 8$ and obtain the lower bound $f_n \ge 3n - 7$ for all $n \ge 5$. Our results and an example provided in the paper support the conjecture that $f_n = \binom{n}{2} - 2$ for each $n \ge 4$. New results regarding the existence of longest fault-free paths with prescribed ends are also proved.

1. INTRODUCTION

When f "faulty" vertices of the same parity are deleted from the $n-$dimensional binary hypercube $\mathcal{Q}_n$, the length of the longest fault-free cycle cannot exceed $2^n - 2f$. A natural question is to determine the maximum integer f_n such that for every set $\mathcal{F}$ of f deleted vertices, with $0 \le f \le f_n$, there exists a cycle of length at least $2^n - 2f$ in the complement of $\mathcal{F}$. Due to applications in parallel computing, this question have been studied by several authors in the last two decades. However, the exact value of f_n for an arbitrary n is yet to be established.

The best result available in the literature is due to J-S. Fu who proved in [F] the following theorem. (We follow the notation and terminology of [Di] unless we explicitly specify otherwise.)

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Theorem 1.1 ([F]). *Let $n \geq 3$ and $\mathcal{F}$ be a set of vertices in $\mathcal{Q}_n$ of cardinality $f \leq 2n-4$. Then there exists a cycle in $\mathcal{Q}_n - \mathcal{F}$ of length at least $2^n - 2f$.*

Fu's result is sharp for $n = 3$ and $n = 4$, and his lower bound of $2n-4$ for f_n was a considerable improvement over previous lower bounds found, for example in [T] and [YTR], where the lower bound did not exceed the dimension. On page 831 of his paper, Fu also made the following remark: "However, it is not easy to prove that a fault-free cycle of length of at least $2^n - 2f$ cannot be embedded in an $n-$cube with f faulty nodes, where $n \geq 5$ and $f \geq 2n-3$."

The main theorem in this paper, with which we improve Fu's theorem (Theorem 1.1), is the following:

Theorem 1.2. *Let $n \geq 5$ and f be integers with $0 \leq f \leq 3n-7$. Then for any set $\mathcal{F}$ of vertices in $\mathcal{Q}_n$ of cardinality f there exists a cycle in $\mathcal{Q}_n - \mathcal{F}$ of length at least $2^n - 2f$.*

As it follows from our theorem, when $n = 5$, for example, it is possible to embed a fault-free cycle of length of at least $2^n - 2f$ in the $n-$cube with f faulty nodes, where $f = 7$ or $f = 8$ which responds to Fu's remark. The following counterexample shows that this is not always possible when $f = 9$. Therefore, we establish that $f_5 = 8$.

Let $n \geq 4$, v be any vertex in $\mathcal{Q}_n$, $\mathcal{S}_2(v)$ be the set of vertices at distance 2 from v, w be any vertex in $\mathcal{S}_2(v)$, and $\mathcal{F} = \mathcal{S}_2(v) \setminus \{w\}$. If there were a cycle of length $2^n - 2|\mathcal{F}|$ in $\mathcal{Q}_n - \mathcal{F}$ it would have to pass through v and through at least two distinct vertices in $\mathcal{S}_2(v)$ which is impossible. This proves that $f_n \leq \binom{n}{2} - 2$ for $n \geq 4$.

Fu's Theorem shows that the upper bound $\binom{n}{2} - 2$ for f_n is sharp for $n = 4$ and our results show that it is sharp for $n = 5$. This motivates the following conjecture:

Conjecture 1.3. *Let $n \geq 4$ and f be integers with $0 \leq f \leq \binom{n}{2} - 2$. Then for any set $\mathcal{F}$ of vertices in $\mathcal{Q}_n$ of cardinality f there exists a cycle in $\mathcal{Q}_n - \mathcal{F}$ of length at least $2^n - 2f$.*

It is clear that if f "faulty" vertices of one parity are deleted from the hypercube $\mathcal{Q}_n$ then the length of the longest path connecting any two non-deleted vertices u, v of that parity cannot exceed $2^n - 2f - 2$. If one of the vertices u, v is even and the other is odd then the length of the longest path connecting them cannot exceed $2^n - 2f - 1$. In [F1] the author proves that if $\mathcal{F}$ is a set of cardinality $f \leq n-2$ then: (1) a path of length $2^n - 2f - 1$ is guaranteed to exist in $\mathcal{Q}_n - \mathcal{F}$ between any two vertices of opposite parity in $\mathcal{Q}_n - \mathcal{F}$; and (2) a path of length $2^n - 2f - 2$ is guaranteed to exist in $\mathcal{Q}_n - \mathcal{F}$ between any two vertices of the same parity in $\mathcal{Q}_n - \mathcal{F}$. In Section 3 we improve these two results by allowing $f \leq n - 1$ in Case (2) without any extra conditions (see Corollary 3.12), and $f \leq n$ in Case (1) with the extra natural condition that at least one neighbor of each end vertex is not the other end vertex and also is not in $\mathcal{F}$ (see Theorems 3.9 and 3.11). In Theorem 3.13 we improve the result mentioned in Case (1) in a different direction. We show that if e is an edge that is not incident to any of the faults and the end vertices then the above mentioned path with length $2^n - 2f - 1$ can be chosen to pass through e. More results about Hamiltonian paths or cycles with prescribed edges in $\mathcal{Q}_n$ can be found in [D] and in [DG].

When additional information regarding the parity of the deleted vertices and of the end vertices is available one should expect to obtain better estimates for the lengths of longest fault-free paths and cycles. A conjecture of Locke [L] states that if the number of deleted vertices from $\mathcal{Q}_n$ is $f = 2k \leq 2n - 4$, with k deleted vertices of each parity, then a fault-free cycle of length $2^n - f$ in $\mathcal{Q}_n$ exists. A proof of this conjecture for $k = 1$ is contained in [LS] and for $k \leq 4$ in [CG2]. More results related to Locke's conjecture can be found in [CG1]. Castañeda and Gotchev have also conjectured that if the number of deleted vertices from $\mathcal{Q}_n$ is $f = 2k + 1 \leq 2n - 5$, with k even and $k + 1$ odd deleted vertices, then for any two non-deleted even vertices there exists a Hamiltonian path (of length $2^n - f - 1$) in the complement of the set of deleted vertices with the two even vertices at the ends [CG2, Conjecture 6.2]. Somehow related to this conjecture is Theorem 4.3 in this article, which states that if $f < n$ vertices of one parity are deleted from $\mathcal{Q}_n$, then for any two given non-deleted vertices of the opposite parity there is a fault-free path of the maximal possible length of $2^n - 2f$ with the two given vertices at the ends. The same theorem also contains a similar result for $f = n$ with the natural additional condition that at least one neighbor of each of the prescribed end vertices is non-deleted.

This article is organized as follows. In Section 2 we introduce the basic definitions and notation used in this paper. Section 3 contains several results regarding the existence of longest path in faulty hypercubes when the ends of the path are prescribed. Section 4 contains some facts regarding longest paths with prescribed ends when additional information about the parity of the faulty vertices and of the end vertices is known. To avoid making the paper too long, in that section we restrict ourselves to consider only the case when all the deleted vertices are of the same parity which is the case that we use later in the paper. Section 5 deals with the case $n = 5$, and Section 6 with the case $n > 5$, of the proof of the main theorem. Some of our proofs use results from the article [CG2] that are summarized in a table in Appendix A. Finally, Appendices B and C contain tables for special cases of the proof of Theorem 3.11.

2. Preliminaries

The $n-$dimensional binary hypercube $\mathcal{Q}_n$ is the graph with a vertex set $\mathcal{V}(\mathcal{Q}_n)$ containing all binary sequences of length n and whose edges are pairs of binary sequences that differ in exactly one position. If $a = (a_1, \ldots, a_n)$ is a vertex in $\mathcal{Q}_n$ then $P_i(a) = a_i$ is the $i-$th component of a. The subgraphs of $\mathcal{Q}_n$ induced by $P_i^{-1}(1)$ and $P_i^{-1}(0)$ will be refered to as *plates*. We call the plates *top* and *bottom* plates and denote them by $\mathcal{Q}_n^{top}$ and $\mathcal{Q}_n^{bot}$. Clearly, each plate is isomorphic to an $(n-1)-$dimensional hypercube. If A is a set of vertices in $\mathcal{Q}_n$ then we set $A^{top} = A \cap \mathcal{V}(\mathcal{Q}_n^{top})$ and $A^{bot} = A \cap \mathcal{V}(\mathcal{Q}_n^{bot})$. A given vertex is called *even* if it has an even number of 1's in its components; otherwise the vertex is called *odd*. In the sequel r, $r_1, \ldots$ represent vertices of one parity in $\mathcal{Q}_n$ that we call *red* and g, $g_1, \ldots$ represent vertices of the opposite parity, that we call *green*. Regarding a pair of vertices we say that the *pair is charged* if the two elements in the pair are of the same parity and that the *pair is neutral* if the two elements are of opposite parity. We call a pair *green* (*red*) if both vertices are green (red).

A *fault* $\mathcal{F}$ in $\mathcal{Q}_n$ is a set of deleted vertices. The *mass* M of a fault $\mathcal{F}$ is the total number of vertices in the fault. Let $r(\mathcal{F})$ be the number of red vertices and $g(\mathcal{F})$ be the number of green vertices in a fault $\mathcal{F}$ of $\mathcal{Q}_n$. The *charge* of a fault $\mathcal{F}$ is the number $C = |r(\mathcal{F}) - g(\mathcal{F})|$. Let also $\mathcal{E}$ be a set of disjoint pairs of vertices of $\mathcal{Q}_n$, $r(\mathcal{E})$ be the number of red pairs in $\mathcal{E}$, and $g(\mathcal{E})$ be the number of green pairs in $\mathcal{E}$. We say that *the set of pairs $\mathcal{E}$ is in balance with the fault $\mathcal{F}$* if all the vertices in the elements of $\mathcal{E}$ are from $\mathcal{Q}_n - \mathcal{F}$ and $r(\mathcal{F}) - g(\mathcal{F}) = g(\mathcal{E}) - r(\mathcal{E})$. A *path covering of a graph G* is a set of vertex disjoint paths that covers all the vertices of G. When the endpoints of a path are of the same parity (color) we say that the path is *charged*; otherwise we say that the path is *neutral*. An obvious necessary condition for a set $\mathcal{E}$ of pairs of vertices to be the set of endpoints of a path covering of $\mathcal{Q}_n - \mathcal{F}$ is that $\mathcal{E}$ be in balance with $\mathcal{F}$.

Definition 2.1. *Let M, C, N, O be nonnegative integers and $\mathcal{F}$ be a fault of mass M and charge C in $\mathcal{Q}_n$. We say that* one can freely prescribe ends for a path covering of $\mathcal{Q}_n - \mathcal{F}$ with N neutral paths and O charged paths *if*

(i) *there exists at least one set $\mathcal{E}$ of disjoint pairs of vertices that is in balance with $\mathcal{F}$ and contains exactly N neutral pairs and O charged pairs; and*
(ii) *for every set $\mathcal{E}$ of disjoint pairs of vertices that is in balance with $\mathcal{F}$ and contains exactly N neutral pairs and O charged pairs there exists a path covering of $\mathcal{Q}_n - \mathcal{F}$ such that the set of pairs of end vertices of the paths in the covering coincides with $\mathcal{E}$.*

It is easy to see that if in $\mathcal{Q}_n$ there exists a fault $\mathcal{F}$ of mass M and charge C, and a set of pairs of vertices $\mathcal{E}$ that is in balance with $\mathcal{F}$ and contains exactly N neutral pairs and O charged pairs, then $2^n \geq M + C + 2N + 2O$.

Let $\mathcal{A}_{M,C,N,O}$ be the set of nonnegative integers m such that

(i) $m \geq \log_2 [M + C + 2N + 2O]$; and
(ii) for every $n \geq m$ and for every fault $\mathcal{F}$ of mass M and charge C in $\mathcal{Q}_n$ one can freely prescribe ends for a path covering of $\mathcal{Q}_n - \mathcal{F}$ with N neutral paths and O charged paths.

We let $[M, C, N, O]$ denote the smallest element in $A_{M,C,N,O}$ if this set is nonempty. All known to us values of $[M, C, N, O]$ are given in a table in Appendix A for easy reference. More information can be found in [CG2].

It is convenient to identify the hypercube $\mathcal{Q}_n$ with the group $\mathbf{Z}_2^n$. We view $\mathcal{Q}_n$ as a Cayley graph with the standard system of generators $\mathbf{S} = \{e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)\}$. An oriented edge in $\mathcal{Q}_n$ is represented by (r, x), where r is the starting vertex and x is an element from the system of generators $\mathbf{S}$. A walk is represented by $(r, \xi; g)$, where r is the initial vertex, g is the end vertex, and ξ is a word with letters from $\mathbf{S}$. If $\xi = x_1 x_2 \cdots x_k$ then the walk $(r, \xi; g)$ is the walk $r, rx_1, rx_1x_2, \ldots, rx_1x_2 \cdots x_k = g$. The algebraic content of a word ξ is the element of $\mathbf{Z}_2^n$ that is obtained by multiplying all the letters of ξ. If r is a vertex, $r\xi$ is the vertex obtained by multiplying r with the algebraic content of ξ. If ξ is the empty word then $r\xi = r$. A walk $(r, \xi; g)$ is a path if no subword of ξ is algebraically equivalent to the identity $(0, 0, \ldots, 0)$. A walk $(r, \xi; g)$ is a cycle if $r = g$, ξ is algebraically equivalent to the identity, and no proper subword of ξ is algebraically equivalent to the identity. Instead of $(r, \xi; r)$ we denote cycles simply by (r, ξ).

We shall use the following notation: ξ^R means the reverse word of ξ; ξ' denotes the word obtained after the last letter is deleted from ξ; ξ^* is the word obtained after the first letter is deleted from ξ; $\varphi(\xi)$ is the first letter of ξ, and $\lambda(\xi)$ is the last letter of ξ. The letter v shall be reserved for steps connecting two plates and the letters $x, y, ...$ shall be reserved to represent steps along the plates. If A is a set of vertices and x is any letter in $\mathbf{S}$ then xA is the set of all vertices of the form ax with $a \in A$.

If $\mathcal{G}$ is a graph and r is a vertex in $\mathcal{G}$ then by $\mathcal{V}(G)$ we denote the set of all vertices of $\mathcal{G}$ and by $\mathcal{N}(r)$ the set of all vertices in $\mathcal{G}$ adjacent to r. If a, b are two vertices in $\mathcal{Q}_n$ then by $d_H(a, b)$ we denote the *Hamming distance* between a and b, i.e. the number of components where a and b differ. By $S_k(a)$ we denote the set of vertices at distance k from a. Finally, for a set A, by $|A|$ we denote the cardinality of A.

3. Longest paths with prescribed ends

In this section we improve all known to us results about longest paths with prescribed ends in hypercubes with faulty vertices. Theorem 3.1 below has its origin in Lemma 2 and Lemma 4 in [F] and is essentially that part of Theorem 2 in Fu's recent paper [F1] that deals with end vertices of the same parity. The part of Fu's result that deals with end vertices of the same parity is contained in Corollary 3.10 and improved in Corollary 3.12 where up to $n-1$ faults are allowed. Theorem 3.1 itself is generalized in two different ways in this section: (1) in Lemma 3.9, and Theorem 3.11, it is extended to allow up to n faults under the mild condition that each end vertex does not get immediately blocked by the fault and the other end vertex; and (2) in Theorem 3.13 where the same $n-2$ faults are allowed but the path is required to pass through an arbitrary prescribed edge that is not incident to the end vertices or any of the faults.

Our proofs in this section take advantage of recent results from [CG2] summarized in Table 1 in Appendix A.

Theorem 3.1. *Let n and f be integers with either $n = 1$ and $f = 0$ or $n \geq 2$ and $0 \leq f \leq n-2$. Let also $\mathcal{F}$ be a set of vertices in $\mathcal{Q}_n$ with cardinality f. Then for any neutral pair of vertices r, g of $\mathcal{Q}_n - \mathcal{F}$ there exists a path in $\mathcal{Q}_n - \mathcal{F}$ of length at least $2^n - 2f - 1$ that goes from r to g.*

Proof. The proof is by induction on f. When $f = 0$ the statement is equivalent to Havel's lemma $[0, 0, 1, 0] = 1$ (see [H] or [CG2, Lemma

3.2]). If $f = 1$ and $n \geq 4$ the statement follows easily from $[2, 0, 1, 0] = 4$ ([CG2, Lemma 3.10]). If $f = 1$ and $n = 3$ the statement can be verified by inspection. Indeed, without loss of generality we can assume that $\mathcal{F}$ consists of a single red vertex r_*; that $r = r_*xy$; and that g is ry, or rz or $rxyz$ for three distinct letters x, y, z. The desired path is then $(r, zxyxz; g)$, $(r, yzxyx; g)$ or $(r, yzyxy; g)$, respectively.

Assume now that $f_\star > 1$ is an integer such that the statement of the lemma is true for all pairs of integers f, n with $0 \leq f < f_\star$ and $n \geq f+2$. We shall prove that the statement is also true for any pair $f_\star, n$ with $n \geq f_\star + 2$.

Let $n \geq f_\star + 2$. Split $\mathcal{Q}_n$ into two plates in such a way that if $f_0 = |\mathcal{V}(\mathcal{Q}_n^{bot}) \cap \mathcal{F}|$ and $f_1 = |\mathcal{V}(\mathcal{Q}_n^{top}) \cap \mathcal{F}|$ then $0 < f_0 \leq f_1 < f_\star$. There are two cases to consider: (1) r and g are on the same plate; (2) r and g are on different plates.

Case (1). We assume that r and g are on the top plate. The same proof works for the case when they are on the bottom plate. By the induction hypothesis there exists a path $(r, \xi; g)$ of length at least $2^{n-1} - 2f_1 - 1$ in $\mathcal{Q}_n^{top} - \mathcal{F}$. Since the path is long enough, we can find words η, θ such that $\xi = \eta\theta$ and neither $r\eta v$ nor $r\eta\varphi(\theta)v$ is in $\mathcal{F}$. By the induction hypothesis again, there is a path $(r\eta v, \mu; r\eta\varphi(\theta))$ of length at least $2^{n-1} - 2f_0 - 1$ in $\mathcal{Q}_n^{bot} - \mathcal{F}$. The path $(r, \eta v\mu v\theta; g)$ is the desired path of length at least $2^n - 2f_\star - 1$ in $\mathcal{Q}_n - \mathcal{F}$.

Case (2). Without loss of generality we can assume that r is in $\mathcal{Q}_n^{top}$ and g is in $\mathcal{Q}_n^{bot}$. Let g_1 be any green vertex in $\mathcal{Q}_n^{top} - \mathcal{F}$ such that $g_1 v \notin \mathcal{F}$. By the induction hypothesis there exist a path $(r, \xi; g_1)$ of length at least $2^{n-1} - 2f_1 - 1$ in $\mathcal{Q}_n^{top} - \mathcal{F}$ and a path $(g_1 v, \eta; g)$ of length at least $2^{n-1} - 2f_0 - 1$ in $\mathcal{Q}_n^{bot} - \mathcal{F}$. The path $(r, \xi v\eta; g)$ is the desired path of length at least $2^n - 2f_\star - 1$ in $\mathcal{Q}_n - \mathcal{F}$. □

Lemma 3.2. *Let $n \geq 3$, $r_\star$ be a red vertex in $\mathcal{Q}_n$, and r and g be a red and a green vertex in $\mathcal{Q}_n - (\mathcal{N}(r_\star) \cup \{r_\star\})$. Then there exists a path $(r, \xi; g)$ of length at least $2^n - 2n - 1$ in $\mathcal{Q}_n - (\mathcal{N}(r_\star) \cup \{r_\star\})$.*

Proof. The statement is obvious for $n = 3$. Let n^* be a positive integer such that the statement of the lemma is true for all n with $3 \leq n < n^*$. We shall prove that the statement is also true for n^*.

Split $\mathcal{Q}_{n^*}$ into two plates so that $r \in \mathcal{V}(\mathcal{Q}_{n^*}^{top})$ and $g \in \mathcal{V}(\mathcal{Q}_{n^*}^{bot})$. We assume that $r_\star \in \mathcal{V}(\mathcal{Q}_{n^*}^{top})$. Similar proof works for the case when $r_\star \in \mathcal{V}(\mathcal{Q}_{n^*}^{bot})$. Let g_1 be a green vertex in $\mathcal{Q}_{n^*}^{top} - (\mathcal{N}(r_\star) \cup \{r_\star\})$. By the induction hypothesis there is a path $(r, \eta; g_1)$ of length at least $2^{n^*-1} - 2(n^*-1) - 1$ in $\mathcal{Q}_{n^*}^{top} - (\mathcal{N}(r_\star) \cup \{r_\star\})$. Observe that there is at most one element of $\mathcal{N}(r_\star)$ in the bottom plate. Therefore, by Theorem 3.1, there is a path $(g_1v, \theta; g)$ of length at least $2^{n^*-1}-2-1$ in $\mathcal{Q}_{n^*}^{bot}-(\mathcal{N}(r_\star)\cup\{r_\star\})$. The path $(r, \eta v\theta; g)$ is the desired path of length at least $2^{n^*}-2n^*-1$ in $\mathcal{Q}_{n^*}-(\mathcal{N}(r_\star)\cup\{r_\star\})$. □

Corollary 3.3. *Let $n \geq 3$, $r_\star$ be a red vertex in $\mathcal{Q}_n$, and g be a green vertex in $\mathcal{Q}_n - \mathcal{N}(r_\star)$. Let also $\mathcal{F} \subset \mathcal{N}(r_\star)$ with $|\mathcal{F}| \leq n-1$. Then there exists a path $(g, \xi; r_\star)$ of length at least $2^n - 2(n-1) - 1$ in $\mathcal{Q}_n - \mathcal{F}$.*

Proof. Let $g_\star \in \mathcal{N}(r_\star) \setminus \mathcal{F}$ and $r \in \mathcal{N}(g_\star) \setminus \{r_\star\}$. By Lemma 3.2, there is a path $(g, \eta; r)$ of length at least $2^n - 2n - 1$ in $\mathcal{Q}_n - (\mathcal{N}(r_\star) \cup \{r_\star\})$. The desired path of length $2^n - 2(n-1) - 1$ in $\mathcal{Q}_n - \mathcal{F}$ is $(g, \eta xy; r_\star)$, where $rx = g_\star$, and $g_\star y = r_\star$. □

Corollary 3.4. *Let $n \geq 3$, and r and $r_\star$ be two red vertices in $\mathcal{Q}_n$. Let also $\mathcal{F} \subset \mathcal{N}(r_\star)$ with $|\mathcal{F}| \leq n-1$ and g be a vertex in $\mathcal{N}(r_\star) \setminus \mathcal{F}$. Then there exists a path $(r, \xi; g)$ of length at least $2^n - 2(n-1) - 1$ in $\mathcal{Q}_n - \mathcal{F}$.*

Proof. Split $\mathcal{Q}_n$ into two plates with $r_\star \in \mathcal{V}(\mathcal{Q}_n^{top})$ and $g \in \mathcal{V}(\mathcal{Q}_n^{bot})$. Observe that $\mathcal{F} \cap \mathcal{V}(\mathcal{Q}_n^{bot}) = \emptyset$. There are two cases to consider: (1) r is on the top plate; (2) r is on the bottom plate.

Case (1). Let g_1 be any green vertex in $\mathcal{Q}_n^{top} - \mathcal{N}(r_\star)$. By Lemma 3.2 there is a path $(r, \eta; g_1)$ of length at least $2^{n-1} - 2(n-1) - 1$ in $\mathcal{Q}_n^{top} - (\mathcal{N}(r_\star) \cup \{r_\star\})$. By Havel's lemma $[0,0,1,0] = 1$ [H] (or by Theorem 3.1) there is a path $(g_1v, \theta; g)$ of length $2^{n-1} - 1$ in $\mathcal{Q}_n^{bot}$. The desired path of length at least $2^n - 2(n-1) - 1$ in $\mathcal{Q}_n - \mathcal{F}$ is $(r, \eta v\theta; g)$.

Case (2). If $n = 3$ the desired path is the Hamiltonian path in $\mathcal{Q}_n^{bot}$ that connects r to g. Let $n \geq 4$. Produce a Hamiltonian path $(r, \eta; g)$ of $\mathcal{Q}_n^{bot}$. There exist words μ and ν such that $\mu\nu = \eta$ and neither $r\mu v$ nor $r\mu\varphi(\nu)v$ is in $\mathcal{N}(r_\star) \cup \{r_\star\}$. By Lemma 3.2 there exists a path $(r\mu v, \zeta; r\mu\varphi(\nu)v)$ of length at least $2^{n-1} - 2(n-1) - 1$ in $\mathcal{Q}_n^{top} - (\mathcal{N}(r_\star) \cup \{r_\star\})$. The desired path of length at least $2^n - 2(n-1) - 1$ in $\mathcal{Q}_n - \mathcal{F}$ is $(r, \mu v\zeta v\nu; g)$. □

Proposition 3.5. *Let $n \geq 2$ and $\mathcal{F}$ be a set of vertices in $\mathcal{Q}_n$ of cardinality $|\mathcal{F}| = n+1$ such that for each $1 \leq i \leq n$ the cardinality of $P_i^{-1}(0) \cap \mathcal{F}$ is either 1 or n. Then $\mathcal{F} = \mathcal{N}(s) \cup \{s\}$ for some $s \in \mathcal{V}(\mathcal{Q}_n)$.*

Proof. The statement is obvious for $n = 2$. Assume now that $n \geq 3$. Let $s = (s_1, \ldots, s_n)$ be such that $s_i = 0$ if $|P_i^{-1}(0) \cap \mathcal{F}| = n$ and $s_i = 1$ otherwise. For every $1 \leq i \leq n$ there exists a unique vertex $a(i) \in \mathcal{F} \setminus \{s\}$ such that $P_i(a(i)) \neq s_i$. Clearly every vertex in $\mathcal{F} \setminus \{s\}$ differs from s at least at one component. Therefore

$$a : \{1, \ldots, n\} \longrightarrow \mathcal{F} \setminus \{s\}$$

defined by the condition $P_i(a(i)) \neq s_i$ is an onto function. Since the cardinality of $\mathcal{F} \setminus \{s\}$ is at least n it follows that a is also one-to-one. Therefore, if $i \neq j$ then $a(i) \neq a(j)$, hence $a(i) \in \mathcal{N}(s)$ for every i. Thus $\mathcal{F} \setminus \{s\} = \mathcal{N}(s)$ and since $|\mathcal{F}| = n+1$ we conclude that $\mathcal{F} = \mathcal{N}(s) \cup \{s\}$. □

Corollary 3.6. *Let $n \geq 2$, $A \subset \{1, 2, \ldots, n\}$, and $\mathcal{F}$ be a set of vertices in $\mathcal{Q}_n$ of cardinality $|\mathcal{F}| = |A|+1$ such that for each $i \in A$ the cardinality of $P_i^{-1}(0) \cap \mathcal{F}$ is either 1 or $|\mathcal{F}| - 1$ and for each $i \notin A$ the cardinality of $P_i^{-1}(0) \cap \mathcal{F}$ is either 0 or $|\mathcal{F}|$. Then $\mathcal{F} = (\mathcal{N}(s) \cap \mathcal{V}(\mathcal{Q}')) \cup \{s\}$, where $\mathcal{Q}'$ is an $|A|-$dimensional subhypercube of $\mathcal{Q}_n$ and s is a vertex in $\mathcal{Q}'$.*

Corollary 3.7. *Let $n \geq 2$ and $\mathcal{F}$ be a set of vertices in $\mathcal{Q}_n$ of cardinality $|\mathcal{F}| = n$. Assume that for some index $1 \leq i_0 \leq n$ the cardinality of $P_{i_0}^{-1}(0) \cap \mathcal{F}$ is either 0 or n and that for all other $1 \leq i \leq n$ the cardinality of $P_i^{-1}(0) \cap \mathcal{F}$ is either 1 or $n-1$. Let $\mathcal{Q}'$ be the hypercube with vertex set $P_{i_0}^{-1}(0)$ if $|P_{i_0}^{-1}(0) \cap \mathcal{F}| = n$ and with vertex set $P_{i_0}^{-1}(1)$ otherwise. Then there exists a vertex $s \in \mathcal{V}(\mathcal{Q}')$ such that $\mathcal{F} = (\mathcal{N}(s) \cap \mathcal{V}(\mathcal{Q}')) \cup \{s\}$.*

Proposition 3.8. *Let $n \geq 2$ and $\mathcal{F}$ be a set of vertices in $\mathcal{Q}_n$ of cardinality $|\mathcal{F}| = n$ such that for each $1 \leq i \leq n$ the cardinality of $P_i^{-1}(0) \cap \mathcal{F}$ is either 1 or $n-1$. Then either $\mathcal{F} = \mathcal{N}(s)$ for some vertex $s \in \mathcal{V}(\mathcal{Q}_n)$ or there exist two vertices $s, t \in \mathcal{V}(\mathcal{Q}_n)$, with $d_H(s,t) = 2$, such that $\mathcal{F}$ consists of s, t and all the vertices of $\mathcal{N}(s)$ except the two vertices that are common neighbors of s and t.*

Proof. The statement is obvious for $n = 2$. Assume now that $n \geq 3$. Let $s = (s_1, \ldots, s_n)$ be such that $s_i = 0$ if $|P^{-1}(0) \cap \mathcal{F}| = n-1$ and $s_i = 1$ otherwise. For every $1 \leq i \leq n$ there exists a unique vertex $a(i) \in \mathcal{F} \setminus \{s\}$ such that $P_i(a(i)) \neq s_i$. Clearly every vertex in $\mathcal{F} \setminus \{s\}$ differs from s at least at one component. Therefore

$$a : \{1, \ldots, n\} \longrightarrow \mathcal{F} \setminus \{s\}$$

defined by the condition $P_i(a(i)) \neq s_i$ is an onto function. If $|\mathcal{F} \setminus \{s\}| = n$ then a is also one-to-one and therefore $\mathcal{F} = \mathcal{N}(s)$. If $|\mathcal{F} \setminus \{s\}| = n-1$ then $s \in \mathcal{F}$, and also, there exist unique $i, j \in \{1, \ldots, n\}$, $i \neq j$, such that $a(i) = a(j) = t$ for some vertex $t \in \mathcal{F}$; hence $d_H(s,t) = 2$. Clearly, $a(k) \in \mathcal{N}(s)$, for $k \neq i, j$, and also the common neighbors of s and t are not in the range of a and therefore, are not in $\mathcal{F}$. □

The following theorem generalizes Lemma 6 in [F].

Theorem 3.9. *Let $n \geq 3$ and $\mathcal{F}$ be a set of vertices of $\mathcal{Q}_n$ of cardinality $|\mathcal{F}| = n-1$. Let also r be a red vertex and g be a green vertex in $\mathcal{Q}_n - \mathcal{F}$*

such that $\mathcal{F} \cup \{r\} \neq \mathcal{N}(g)$ *and* $\mathcal{F} \cup \{g\} \neq \mathcal{N}(r)$. *Then there exists a path* $(r, \xi; g)$ *of length at least* $2^n - 2(n-1) - 1$ *in* $\mathcal{Q}_n - \mathcal{F}$.

Proof. If $\mathcal{F} \subset \mathcal{N}(r)$ or if $\mathcal{F} \subset \mathcal{N}(g)$ then the statement follows from Corollary 3.3. Therefore we can assume that $\mathcal{F} \not\subset \mathcal{N}(r)$ and $\mathcal{F} \not\subset \mathcal{N}(g)$.

The statement is easy to verify for $n = 3$. Assume now that $n \geq 4$. Split Q_n into two plates such that r and g are in different plates. Without loss of generality we can assume that $r \in \mathcal{V}(\mathcal{Q}_n^{top})$. We can also assume that $0 \leq f_0 = |\mathcal{F}^{bot}| \leq f_1 = |\mathcal{F}^{top}|$. There are two cases to consider: (1) $f_1 \leq n-2$; and (2) $f_1 = n-1$.

Case (1). Let g_1 be any green vertex in $\mathcal{Q}_n^{top} - \mathcal{F}$ such that $g_1 v \notin \mathcal{F}$. According to Theorem 3.1 there exists a path $(r, \eta; g_1)$ of length at least $2^{n-1} - 2f_1 - 1$ in $\mathcal{Q}_n^{top} - \mathcal{F}$, and a path $(g_1 v, \theta; g)$ of length at least $2^{n-1} - 2f_0 - 1$ in $\mathcal{Q}_n^{bot} - \mathcal{F}$. The path $(r, \eta v \theta; g)$ is the desired path of length at least $2^n - 2(n-1) - 1$ in $\mathcal{Q}_n - \mathcal{F}$.

Case (2). Let $n = 4$. There exists a green vertex $g_1 = rx \in \mathcal{V}(\mathcal{Q}_n^{top}) - \mathcal{F}$. Let $(g_1 v, \xi; g)$ be a Hamiltonian path in $\mathcal{Q}_n^{bot}$. Then the path $(r, xv\xi; g)$ is a path with length nine, as it is required.

To prove our claim for each $n \geq 5$ we proceed by induction on n. Assume that $N \geq 5$ is any integer and that the statement is true for all integers n with $4 \leq n < N$. We shall prove that the statement is also true for N.

Let a be any vertex in $\mathcal{F} \setminus \mathcal{N}(r)$ and g_1 be any green vertex in $\mathcal{Q}_N^{top} - \mathcal{F}$ such that $(\mathcal{F} \setminus \{a\}) \cup \{g_1\} \neq \mathcal{N}(r) \cap \mathcal{V}(\mathcal{Q}_N^{top})$ and $(\mathcal{F} \setminus \{a\}) \cup \{r\} \neq \mathcal{N}(g_1) \cap \mathcal{V}(\mathcal{Q}_N^{top})$. By the induction hypothesis there is a path $(r, \eta; g_1)$ of length at least $2^{N-1} - 2(N-2) - 1$ in $\mathcal{Q}_N^{top} - (\mathcal{F} \setminus \{a\})$. There are three subcases: (i) a is not in the path $(r, \eta; g_1)$; (ii) $a = r\eta'$ (iii) a is in the path $(r, \eta; g_1)$ but $a \neq r\eta'$.

Subcase (i). By Havel's lemma $[0,0,1,0] = 1$ [H] there is a path $(g_1 v, \theta; g)$ of length $2^{N-1} - 1$ in $\mathcal{Q}_N^{bot}$. The path $(r, \eta v \theta; g)$ is a path of length at least $2^N - 2(N-2) - 1$ in $\mathcal{Q}_N - \mathcal{F}$.

Subcase (ii). By Havel's lemma $[0,0,1,0] = 1$ [H] there is a path $(r\eta'' v, \theta; g)$ of length $2^{N-1} - 1$ in $\mathcal{Q}_N^{bot}$. The path $(r, \eta'' v \theta; g)$ is a path of length at least $2^N - 2(N-1) - 1$ in $\mathcal{Q}_N - \mathcal{F}$.

Subcase (iii). There exist words μ and ν such that $\eta = \mu\nu$ and $a = r\mu$. Since $a \neq r\eta'$ the length of ν is at least two and since a is not a neighbor of r the length of μ is also at least two. Clearly g is not contained in at least one of the two disjoint sets $A_1 = \{r\mu'' v, g_1(\nu^R)' v\}$ or $A_2 = \{r\mu' v, g_1(\nu^R)'' v\}$.

Let us assume first that $g \notin A_1$. Since $[0,0,2,0] = 2$ (see [D] or [CG2, Lemma 3.3]), there is a $2-$path covering $(r\mu''v, \theta; g_1(\nu^R)'v)$, $(g_1v, \zeta; g)$ of $\mathcal{Q}_N^{bot}$. Therefore, the path $(r, \mu''v\theta v\nu^* v\zeta; g)$ is the desired path of length at least $2^N - 2(N-1) - 1$ in $\mathcal{Q}_N - \mathcal{F}$.

Let us assume now that $g \in A_1$, hence $g \notin A_2$. If the length of ν is more than two then $g_1(\nu^R)'' \neq g_1$, and since $[0,0,2,0] = 2$ (see [D] or [CG2, Lemma 3.3]), there is a $2-$path covering $(r\mu'v, \theta; g_1(\nu^R)''v)$, $(g_1v, \zeta; g)$ of $\mathcal{Q}_N^{bot}$. The path $(r, \mu'v\theta v\nu^{**}v\zeta; g)$ is the desired path of length at least $2^N - 2(N-1) - 1$ in $\mathcal{Q}_N - \mathcal{F}$. If the length of ν is two then $g = r\eta'v$. Let $r_1 \in \mathcal{Q}_N^{bot}$ be a red vertex different from g_1v. Since $N \geq 5$ and $[2,0,1,0] = 4$ ([CG2, Lemma 3.10]) there exists a Hamiltonian path $(r\mu'v, \theta; g_1v)$ for $\mathcal{Q}_N^{bot} - \{r_1, g\}$ of length $2^{N-1} - 3$. The path $(r, \mu'v\theta v\varphi(\nu^R)v; g)$ is the desired path of length at least $2^N - 2(N-1) - 1$ in $\mathcal{Q}_N - \mathcal{F}$. □

Corollary 3.10. *Let $n \geq 2$ and f be integers with $0 \leq f \leq n-2$ and $\mathcal{F}$ be a set of vertices in $\mathcal{Q}_n$ of cardinality f. Then for any pair of green vertices g_1, g_2 in $\mathcal{Q}_n - \mathcal{F}$ there exists a path of length at least $2^n - 2f - 2$ in $\mathcal{Q}_n - \mathcal{F}$ that goes from g_1 to g_2.*

Proof. The statement is obvious if $n = 2$. Let $n \geq 3$ and $r = g_2x$ be a neighbor of g_2 in $\mathcal{Q}_n$ such that $\mathcal{F} \cup \{g_1, g_2\} \neq \mathcal{N}(r)$. Obviously $\mathcal{F} \cup \{r\} \cup \{g_2\} \neq \mathcal{N}(g_1)$. Therefore, by Theorem 3.1, in the case when $f \leq n-3$, and by Theorem 3.9, in the case when $f = n-2$, there exists a path $(g_1, \xi; r)$ of length at least $2^n - 2(f+1) - 1$ in $\mathcal{Q}_n - (\mathcal{F} \cup \{g_2\})$. The path $(g_1, \xi x; g_2)$ is the desired path of length at least $2^n - 2f - 2$ in $\mathcal{Q}_n - \mathcal{F}$. □

Theorem 3.11. *Let $n \geq 3$ and $\mathcal{F}$ be a set of vertices in $\mathcal{Q}_n$ of cardinality $|\mathcal{F}| = n$. Let also r be a red vertex and g be a green vertex in $\mathcal{Q}_n - \mathcal{F}$ such that $\mathcal{N}(g) \not\subset \mathcal{F} \cup \{r\}$ and $\mathcal{N}(r) \not\subset \mathcal{F} \cup \{g\}$. Then there exists a path of length at least $2^n - 2n - 1$ in $\mathcal{Q}_n - \mathcal{F}$ that goes from r to g.*

Proof. The statement is obvious for $n = 3$. We give a separate proof for $n = 4$ and use mathematical induction for $n \geq 5$.

Let $n = 4$ and let $\mathcal{F}$ be any fault of mass 4 in $\mathcal{Q}_4$. We have to exhibit paths of length at least $2^n - 2n - 1 = 7$ in $\mathcal{Q}_4 - \mathcal{F}$ that go from g to r. There are three cases to consider: (1) there is a splitting of $\mathcal{Q}_4$ such that $f_0 = |\mathcal{F}^{bot}| = 0$, and $f_1 = |\mathcal{F}^{top}| = 4$; (2) there is a splitting of $\mathcal{Q}_4$ such that $f_0 = |\mathcal{F}^{bot}| = 1$, and $f_1 = |\mathcal{F}^{top}| = 3$; and (3) for every splitting of $\mathcal{Q}_4$ there are exactly two deleted vertices on each plate.

Case (1). The four deleted vertices are on $\mathcal{Q}_4^{top}$.

Subcase (i). r and g are on $\mathcal{Q}_4^{top}$.

Let $(rv, \eta; gv)$ be a Hamiltonian path of $\mathcal{Q}_4^{bot}$. The desired path of length at least 7 in $\mathcal{Q}_4 - \mathcal{F}$ is $(r, v\eta v; g)$.

Subcase (ii). r and g are on $\mathcal{Q}_4^{bot}$.

A Hamiltonian path of $\mathcal{Q}_4^{bot}$ that goes from r to g does the job in this case.

Subcase (iii). r is on $\mathcal{Q}_4^{top}$ and g is on $\mathcal{Q}_4^{bot}$.

If $rv \neq g$ then there exists a path $(rv, \eta; g)$ of length six in $\mathcal{Q}_4^{bot}$. Then the path $(r, v\eta; g)$ does the job in this case.

Now, let $rv = g$. Since $\mathcal{N}(r) \not\subset \mathcal{F} \cup \{g\}$, there exists a letter $x \neq v$ such that $rx \notin \mathcal{F}$. Let $(rxv, \eta; g)$ be a Hamiltonian path of $\mathcal{Q}_4^{bot}$. The desired path of length at least 7 in $\mathcal{Q}_4$ is $(r, xv\eta; g)$.

Subcase (iv). g is on $\mathcal{Q}_4^{top}$ and r is on $\mathcal{Q}_4^{bot}$.

This subcase is equivalent to *Subcase (iii).*

Case (2). There are exactly three deleted vertices on the top plate.

Without loss of generality we may assume that the deleted vertex on $\mathcal{Q}_4^{bot}$ is a red vertex r_1.

Subcase (i). r and g are on $\mathcal{Q}_4^{top}$.

If $gv \neq r_1$, then by Theorem 3.1 there is a path $(rv, \xi; gv)$ of length 5 in $\mathcal{Q}_4^{bot} - \{r_1\}$. The desired path of length at least 7 in $\mathcal{Q}_4$ is $(r, v\xi v; g)$.

If $gv = r_1$, then there is a letter $x \neq v$ such that $gx \notin \mathcal{F}$. There is also a path $(rv, \eta; gxv)$ that is Hamiltonian in $\mathcal{Q}_4^{bot} - \{r_1\}$. The desired path of length at least 7 in $\mathcal{Q}_4$ is $(r, v\eta vx; g)$.

Subcase (ii). r and g are on $\mathcal{Q}_4^{bot}$.

For any choice of r, g, and r_1 there exist two different $r - g$ paths $P_1 : (r, x_1x_2x_3x_4x_5; g)$ and $P_2 : (r, y_1y_2y_3y_4y_5; g)$ of length five in $\mathcal{Q}_4^{bot} - r_1$ such that rx_1 is not a vertex in P_2 and ry_1 is not a vertex in P_1 (the three essentially different cases are shown in Table 2 in Appendix B). For convenience let $\xi_i = x_1 \ldots x_i$, $\eta_i = y_1 \ldots y_i$ for $i = 1, \ldots, 5$ and let $\xi_0 = \eta_0 = \emptyset$ be empty words. If $r\xi_{i-1}v$ and $r\xi_i v$ are not faulty vertices for some i then the desired path is $(r, \xi_{i-1}vx_ivx_{i+1}\ldots x_5; g)$. The same applies if ξ is replaced by η. One of the two previous situations will happen if no more than two faulty vertices from the top plate project down to vertices of P_1 or P_2. If, on the other hand, all the three faulty vertices on the top plate project down to vertices in each of the paths P_1, P_2 then rx_1v cannot be faulty for rx_1 is not a vertex in P_2. The only way that no two consecutive vertices of P_1 are projections of fault-free vertices from the top plate is if the faulty vertices of the top plate are $rv, rx_1x_2v, rx_1x_2x_3x_4v$. Assuming, without loss of generality, that $r_1 = rxy$ and taking P_1 to be the first path in Table 2 in Appendix B, we have the following three cases: If $P_1 = (r, zxyxz; g)$ then the desired $r-g$ path in $Q_4 - \mathcal{F}$ is $(r, zxyvzxv; g)$; if $P_1 = (r, yzxyx; g)$ then the desired $r-g$ path in $Q_4 - \mathcal{F}$ is $(r, yvxzvyx; g)$; finally if $P_1 = (r, yzyxy; g)$ then the desired $r-g$ path in $Q_4 - \mathcal{F}$ is $(r, xvyxvzyxy; g)$.

Subcase (iii). r is on $\mathcal{Q}_4^{top}$ and g is on $\mathcal{Q}_4^{bot}$.

If $rv \neq g$ then there exists a Hamiltonian path $(rv, \xi; g)$ for $\mathcal{Q}_4^{bot} - \{r_1\}$ which clearly has length 6. Then $(r, v\xi; g)$ is the desired path with length 7.

Now let $rv = g$. Then $\mathcal{N}(r)^{top} \not\subset \mathcal{F}$. If there exists $g_1 = rx \in \mathcal{N}(r)^{top} \setminus \mathcal{F}$ such that $g_1v \neq r_1$ then there exists a path $(g_1v, \eta; g)$ of length five in $\mathcal{Q}_4^{bot} - r_1$. The path $(r, xv\eta; g)$ is the desired path of length seven in $\mathcal{Q}_4 - \mathcal{F}$. If for every $g_1 = rx \in \mathcal{N}(r)^{top} \setminus \mathcal{F}$ we have $g_1v = r_1$ then such g_1 must be unique; hence $\mathcal{N}(r)^{top} \subset \mathcal{F} \cup \{r_1v\}$. Therefore at least two of the deleted vertices in the top plate are green. Thus g_1 has a red neighbor $r_2 = g_1y \in \mathcal{Q}_4^{top} - \mathcal{F}$ which is different from r. Since there exists a Hamiltonian path $(r_2v, \xi; g)$ for $\mathcal{Q}_4^{bot} - \{r_1\}$ with length 6, the desired path with length at least 7 is $(r, xyv\xi; g)$.

Subcase (iv). g is on $\mathcal{Q}_4^{top}$ and r is on $\mathcal{Q}_4^{bot}$.

If $\mathcal{N}(g)^{top} \not\subset \mathcal{F}$, we select a vertex $r_2 \in \mathcal{Q}_n^{top}$ such that $\mathcal{N}(r_2)^{top} \not\subset \mathcal{F}$. Let $(g, \xi; r_2)$ be a path of length at least one in $\mathcal{Q}_4^{top} - \mathcal{F}$, and $(r_2v, \eta; r)$ be a path of length five in $\mathcal{Q}_4^{bot} - \{r_1\}$. The desired path of length at least seven in $\mathcal{Q}_4 - \mathcal{F}$ is $(g, \xi v\eta; r)$.

If $\mathcal{N}(g)^{top} \subset \mathcal{F}$, then (1) $gv \neq r$ and $gv \neq r_1$; (2) all the deleted vertices on the top plate are red; (3) each pair of green vertices on the top plate, not containing g, can be connected by a path of length two in $\mathcal{Q}_4^{top} - \mathcal{F}$; and (4) there is a red vertex $r_2 \neq r, r_1, gv$ in $\mathcal{Q}_4^{bot}$ and a path $(gv, \eta; r_2)$ of length two in $\mathcal{Q}_4^{bot} - \{r, r_1\}$. Let $(r_2v, \theta; rv)$ be a path of length two in $\mathcal{Q}_4^{top} - \mathcal{F}$. The desired path of length seven in $\mathcal{Q}_4 - \mathcal{F}$ is $(g, v\eta v\theta v; r)$.

Case (3). For every splitting of $\mathcal{Q}_4$ there are exactly two deleted vertices on each plate.

Since there is a splitting that puts r and g on the same plate, we can assume, without loss of generality, that r and g are on the top plate. Up to isomorphism, there are five different types of faults that satisfy the property of this case. It is possible to inspect each case to see that the desired path of length at least seven always exists. For the benefit of the reader we have arranged all possible cases in a table in Appendix C. This completes the case $n = 4$.

Let $n \geq 5$. If $\mathcal{F} \subset \mathcal{N}(s) \cup \{s\}$ for some vertex s, the statement follows from Lemma 3.2. So we assume that $\mathcal{F} \not\subset \mathcal{N}(s) \cup \{s\}$ for any $s \in \mathcal{Q}_n$, and proceed by induction.

It follows from the above assumption, Corollary 3.7, and Proposition 3.8 that there are two possibilities: (1) there is a splitting of $\mathcal{Q}_n$ into two plates such that $2 \leq f_0 = |\mathcal{F}^{bot}| \leq |\mathcal{F}^{top}| = f_1 \leq n-2$; and (2) there exist a splitting of $\mathcal{Q}_n$ and two vertices $s \in \mathcal{F}^{top}$ and $t \in \mathcal{F}^{bot}$ such that $d_H(s,t) = 2$, $\mathcal{F}^{top}$ consists of s and $n-2$ neighbors of s, and $\mathcal{F}^{bot} = \{t\}$.

For every one of these two cases we shall consider three subcases: (i) r and g are on the top plate; (ii) r and g are on the bottom plate; (iii) r and g are on different plates.

Case (1). There is a splitting of $\mathcal{Q}_n$ into two plates such that $2 \leq f_0 = |\mathcal{F}^{bot}| \leq |\mathcal{F}^{top}| = f_1 \leq n-2$.

Subcase (i). First we consider the extreme case where r is adjacent to g and either (a) $|\mathcal{N}(r) \cap \mathcal{F}^{top}| = n-2$ or (b) $|\mathcal{N}(g) \cap \mathcal{F}^{top}| = n-2$. The cases (a) and (b) are symmetric, so we shall discuss here only case (a). Let r_1 be a red vertex in the top plate such that $r_1 v \notin \mathcal{F}$. By Theorem 3.9 there is a path $(g, \eta; r_1)$ of length at least $2^{n-1} - 2(n-2) - 1$ in $\mathcal{Q}_n^{top} - \mathcal{F}$. By Corollary 3.10 there is a path $(r_1 v, \theta; rv)$ of length $2^{n-1} - 2 \cdot 2 - 2$ in $\mathcal{Q}_n^{bot} - \mathcal{F}$. The desired path of length at least $2^n - 2n - 1$ in $\mathcal{Q}_n - \mathcal{F}$ is $(g, \eta v \theta v; r)$.

If the extreme case above does not happen then $\mathcal{N}(r)^{top} \not\subset \mathcal{F}^{top} \cup \{g\}$ and $\mathcal{N}(g)^{top} \not\subset \mathcal{F}^{top} \cup \{r\}$. Therefore, by Theorem 3.1 or by Theorem 3.9, there exists a path $(r, \xi; g)$ of length at least $2^{n-1} - 2f_1 - 1$ in $\mathcal{Q}_n^{top} - \mathcal{F}$. Since $2^{n-1} - 2f_1 - 1 > 2f_0 + 1$, there exist two words η and θ such that $\xi = \eta\theta$ and neither $r\eta v$ nor $r\eta\varphi(\theta)v$ is in $\mathcal{F}$. By Theorem 3.1, there exists a path $(r\eta v, \mu; r\eta\varphi(\theta)v)$ of length at least $2^{n-1} - 2f_0 - 1$ in $\mathcal{Q}_n^{bot} - \mathcal{F}$. The desired path of length at least $2^n - 2n - 1$ in $\mathcal{Q}_n - \mathcal{F}$ is $(r, \eta v \mu v \theta^*)$.

Subcase (ii). The proof is similar to the proof of *Subcase (i).*

Subcase (iii). Without loss of generality we can assume that r is on the top plate and g is on the bottom plate. Let g_1 be a green vertex on the top plate such that $g_1 v \notin \mathcal{F}$. By Theorem 3.1 there exists a path $(r, \xi; g_1)$ of length at least $2^{n-1} - 2f_1 - 1$ in $\mathcal{Q}_n^{top} - \mathcal{F}$ and a path $(g_1 v, \eta; g)$ of length at least $2^{n-1} - 2f_0 - 1$ in $\mathcal{Q}_n^{bot} - \mathcal{F}$. The path $(r, \xi v \eta; g)$ is the desired path of length at least $2^n - 2n - 1$ in $\mathcal{Q}_n - \mathcal{F}$.

Case (2). There exist a splitting of $\mathcal{Q}_n$ and two vertices $s \in \mathcal{F}^{top}$ and $t \in \mathcal{F}^{bot}$ such that $d_H(s,t) = 2$, $\mathcal{F}^{top}$ consists of s and $n-2$ neighbors of s, and $\mathcal{F}^{bot} = \{t\}$.

Subcase (i). By the induction hypothesis, there exists a path $(r, \xi; g)$ of length at least $2^{n-1} - 2(n-1) - 1$ in $\mathcal{Q}_n^{top} - \mathcal{F}$. Since the path is long enough, there exist words μ and ν such that $\xi = \mu\nu$, $r\mu v \neq t$, and $r\mu\varphi(\nu)v \neq t$. By Theorem 3.1, there exists a path $(r\mu v, \eta; r\mu\varphi(\nu)v)$ of length at least $2^{n-1} - 2 - 1$ in $\mathcal{Q}_n^{bot} - \{t\}$. The desired path of length at least $2^n - 2n - 1$ in $\mathcal{Q}_n - \mathcal{F}$ is $(r, \mu v\eta v\nu^*; g)$.

Subcase (ii). The proof is similar to the proof of *Subcase (i).*

Subcase (iii). Without loss of generality we can assume that r is on the top plate and g is on the bottom plate. Let g_1 be a green vertex on the top plate such that $g_1 v \notin \mathcal{F}$. We can also assume that g_1 is such that $\mathcal{N}(r)^{top} \not\subset \mathcal{F}^{top} \cup \{g_1\}$ and $\mathcal{N}(g_1)^{top} \not\subset \mathcal{F}^{top} \cup \{r\}$. Therefore we can apply the induction hypothesis and find a path $(r, \xi; g_1)$ of length at least $2^{n-1} - 2(n-1) - 1$ in $\mathcal{Q}_n^{top} - \mathcal{F}$. By Theorem 3.1, there is a path $(g_1 v, \eta; g)$ of length at least $2^{n-1} - 2 - 1$ in $\mathcal{Q}_n^{bot} - \{t\}$. The desired path of length at least $2^n - 2n - 1$ in $\mathcal{Q}_n - \mathcal{F}$ is $(r, \xi v\eta; g)$. □

Now we can improve Corollary 3.10 to allow up to $n-1$ faults.

Corollary 3.12. *Let $n \geq 2$ and f be integers with $0 \leq f \leq n-1$ and $\mathcal{F}$ be a set of vertices in $\mathcal{Q}_n$ of cardinality f. Then for any pair of green vertices g_1, g_2 in $\mathcal{Q}_n - \mathcal{F}$ there exists a path of length at least $2^n - 2f - 2$ in $\mathcal{Q}_n - \mathcal{F}$ that goes from g_1 to g_2.*

Proof. If $0 \leq f \leq n-2$ then the claim is contained in Corollary 3.10. Let $f = n-1$. If $n = 2$ the claim is obvious. If $n \geq 3$ then there exists $r \in \mathcal{N}(g_2)$ such that $\mathcal{N}(g_1) \not\subset \mathcal{F} \cup \{r\}$ and $\mathcal{N}(r) \not\subset \mathcal{F} \cup \{g_1\}$. Then it follows from Theorem 3.11 that there exists a path $(g_1, \eta; r)$ of length at least $2^{n-1} - 2n - 1$ in $\mathcal{Q}_n - \{\mathcal{F} \cup \{g_2\}\}$. If $x \in \mathbf{S}$ is such that $rx = g_2$, then $(g_1, \eta x; g_2)$ is the desired path in $\mathcal{Q}_n - \mathcal{F}$ with length at least $2^{n-1} - 2(n-1) - 2$. □

Sometimes it is convenient to prescribe not only the ends of the faultless path but also a specific edge through which the path is forced to pass. That is the case covered in the following lemma that is an important ingredient in the proof of the main theorem.

Theorem 3.13. *Let $n \geq 3$ and f be integers with $0 \leq f \leq n-2$. Let also $\mathcal{F}$ be a set of vertices in $\mathcal{Q}_n$ of cardinality f. Then for any neutral pair of vertices r, g in $\mathcal{Q}_n - \mathcal{F}$ and for every edge $e = \{r_1, g_1\}$*

not incident to any of the vertices in $\mathcal{F} \cup \{r, g\}$*, there exists a path of length at least* $2^n - 2f - 1$ *in* $\mathcal{Q}_n - \mathcal{F}$ *that goes from* r *to* g *and passes through* e*.*

Proof. In the first part of the proof we justify the claim for all the pairs f, n with $f \leq 2$ and $n \geq f + 2$. We let z be the element in $\mathbf{S}$ such that $g_1 = r_1 z$.

The claim is obvious for $f = 0$ and $n = 2$. For $f = 0$ and any $n \geq 3$ the claim of the lemma follows readily from the fact that $[0, 0, 2, 0] = 2$ ([CG2, Lemma 3.3]). For $f = 1$ and $n = 3$ the claim can be verified by inspection. For $f = 1$ and any $n \geq 4$, the claim of the lemma follows from the fact that $[2, 0, 2, 0] = 4$ ([CG2, Lemma 5.6]). Indeed, let r_2, g_2 be a red and a green vertices in $\mathcal{Q}_n - \{r, g, r_1, g_1\}$ such that $\mathcal{F} \subset \{r_2, g_2\}$. Let $(r, \xi; g_1)$, $(r_1, \eta; g)$ be a two path covering of $\mathcal{Q}_n - \{r_2, g_2\}$. The path $(r, \xi z \eta; g)$ is the desired path of length $2^n - 2f - 1$ in $\mathcal{Q}_n - \mathcal{F}$. In a similar way, if $f = 2$, $n \geq 4$, and the two vertices in $\mathcal{F}$ are of opposite parity, the statement of the lemma follows again from $[2, 0, 2, 0] = 4$.

To finish the first part of the proof we just need to consider the case when $f = 2, n \geq 4$, and the two vertices in $\mathcal{F}$ are of the same parity. Clearly, in this case we can split $\mathcal{Q}_n$ into two plates such that

(i) The edge e is contained in $\mathcal{Q}_n^{top}$; and
(ii) $f_0 = f_1 = 1$, where $f_0 = |\mathcal{F}^{bot}|$ and $f_1 = |\mathcal{F}^{top}|$.

There are three cases to consider: (1) r and g are on the top plate; (2) r and g are on the bottom plate; (3) r and g are on different plates. The proof of each of these cases is similar to the proof of the corresponding cases in the second part of the proof below.

For the second part of the proof we assume that $N \geq 5$ is a positive integer such that the statement is true for all pairs of integers n, f with $4 \leq n < N$ and $2 \leq f \leq n - 2$. We shall prove that the statement is also true for all pairs N, f with $3 \leq f \leq N - 2$.

Since $f \geq 3$, we can split $\mathcal{Q}_N$ into two plates such that

(i) The edge e is contained in $\mathcal{Q}_N^{top}$; and
(ii) $1 \leq f_0, f_1 \leq N - 3$, where $f_0 = |\mathcal{F}^{bot}|$ and $f_1 = |\mathcal{F}^{top}|$.

There are three cases to consider: (1) r and g are on the top plate; (2) r and g are on the bottom plate; (3) r and g are on different plates.

Case (1). By the induction hypothesis, there exists a path $(r, \xi; g)$ of length at least $2^{N-1} - 2f_1 - 1$ in $\mathcal{Q}_N^{top} - \mathcal{F}$ that passes through the edge e. Since $2^{N-1} - 2f_1 - 1 > 2f_0 + 3$, there exist words η and θ such that $\xi = \eta\theta$ and none of the vertices $r\eta$, $r\eta\varphi(\theta)$, $r\eta v$, or $r\eta\varphi(\theta)v$ is in $\mathcal{F} \cup \{r_1, g_1\}$. By Theorem 3.1, there exists a path $(r\eta v, \mu; r\eta\varphi(\theta)v)$ of length at least $2^{N-1} - 2f_0 - 1$ in $\mathcal{Q}_N^{bot} - \mathcal{F}$. The desired path of length at least $2^N - 2f - 1$ in $\mathcal{Q}_n - \mathcal{F}$ that passes through the edge e is $(r, \eta v\mu v\theta^*)$.

Case (2). By Theorem 3.1, there exists a path $(r, \xi; g)$ of length at least $2^{N-1} - 2f_0 - 1$ in $\mathcal{Q}_N^{bot} - \mathcal{F}$. Since $2^{N-1} - 2f_0 - 1 > 2f_1 + 3$, there exist words η and θ such that $\xi = \eta\theta$ and neither $r\eta v$ nor $r\eta\varphi(\theta)$ is in $\mathcal{F} \cup \{r_1, g_1\}$. By the induction hypothesis there exists a path $(r_1\eta v, \mu; r\eta\varphi(\theta)v)$ of length at least $2^{N-1} - 2f_1 - 1$ in $\mathcal{Q}_N^{top} - \mathcal{F}$ that passes through the edge e. The desired path of length at least $2^N - 2f - 1$ in $\mathcal{Q}_N - \mathcal{F}$ is $(r, \eta v\mu v\theta^*; g)$.

Case (3). Without loss of generality we can assume that r is on the top plate and g is on the bottom plate. Let g_2 be any green vertex in $\mathcal{Q}_N^{top} - (\mathcal{F} \cup \{g_1, r_1\})$ such that $g_2 v$ is not in $\mathcal{F}$. By the induction hypothesis, there is a path $(r, \xi; g_2)$ of length at least $2^{N-1} - 2f_1 - 1$ in $\mathcal{Q}_N^{top} - \mathcal{F}$ that passes through the edge e. By Theorem 3.1, there exists a path $(g_2 v, \mu; g)$ of length at least $2^{N-1} - 2f_0 - 1$ in $\mathcal{Q}_N^{bot} - \mathcal{F}$. The desired path of length at least $2^N - 2f - 1$ in $\mathcal{Q}_N - \mathcal{F}$ that passes through the edge e is $(r, \xi v\mu; g)$. □

4. A special case of $\mathcal{F}$

In this section we show that a better estimate for the length of the maximal path in $\mathcal{Q}_n - \mathcal{F}$ can be obtained if all of the deleted vertices in $\mathcal{F}$ have the same parity and the terminals of the path are of the opposite parity.

The following two lemmas discuss the case of $\mathcal{Q}_4$ and serve as a preparation to the proof of Theorem 4.3 below and the case $n = 5$ of the main theorem that is covered in the next section. We do not consider here the cases $|\mathcal{F}| = 1$ and $|\mathcal{F}| = 2$ when $n = 4$ since they are covered by $[1, 1, 0, 1] = 2$ (see [LW] or [CG2, Lemma 3.5]) and by $[3, 1, 0, 1] = 4$ ([CG2, Lemma 3.17]), respectively.

Lemma 4.1. *Let $\mathcal{F} = \{r_1, r_2, r_3\} \subset \mathcal{V}(\mathcal{Q}_4)$, and g_1 and g_2 be two green vertices in $\mathcal{Q}_4 - \mathcal{F}$. Then there exists a path $(g_1, \eta; g_2)$ in $\mathcal{Q}_4 - \mathcal{F}$ which has the maximal possible length ten.*

Proof. Without loss of generality we can assume that $\mathcal{F}^{top} = \{r_1, r_2\}$ and $\mathcal{F}^{bot} = \{r_3\}$. We have to consider three cases: (1) the green terminals are on the top plate; (2) the green terminals are on the bottom plate; and (3) the green terminals are in separate plates.

Case (1). It is easy to see that there are letters $x, y \in \mathbf{S}$ such that $g_1xv, g_2yv \in Q_4^{bot} - \mathcal{F}$. Let $(g_1xv, \xi; g_2yv)$ be a Hamiltonian path of $\mathcal{Q}_4^{bot} - \{r_3\}$. Then $(g_1, xv\xi vy; g_2)$ is the desired path of length ten in $\mathcal{Q}_4 - \mathcal{F}$.

Case (2). Let $(g_1, \xi; g_2)$ be a Hamiltonian path of $\mathcal{Q}_4^{top} - \{r_3\}$. There are words μ, ν such that $\xi = \mu\nu$ and $g_1\mu v, g_1\mu' v$ are in the two dimensional plane parallel to the plane that contains $\{r_2, r_3\}$ in the top plate. Obviously, there is a path $(g_1\mu' v, \eta; g_1\mu v)$ of length three in $\mathcal{Q}_4^{top} - \mathcal{F}$. Then $(g_1, \mu' v\eta v\nu; g_2)$ is the desired path of length ten in $\mathcal{Q}_4 - \mathcal{F}$.

Case (3). Without loss of generality we can assume that g_1 is on the top plate and g_2 is on the bottom plate.

If g_1 is adjacent to both r_1 and r_2 then there is a path $(g_1, \xi; g_3)$ of length four in $\mathcal{Q}_4^{top} - \mathcal{F}$, where g_3 is in $\mathcal{Q}_4^{top} - \mathcal{F}$ and is such that $g_3v \neq r_3$. By Theorem 3.1 there is a path $(g_3v, \eta; g_2)$ of length five in $\mathcal{Q}_4^{bot} - \mathcal{F}$. The path $(g_1, \xi v\eta; g_2)$ is the desired path of length ten in $\mathcal{Q}_4 - \mathcal{F}$.

If g_1 is not adjacent to one of the deleted vertices of the top plate, then there is a path $(g_1, \xi; r_4)$ of length three in $\mathcal{Q}_4^{top} - \mathcal{F}$, where r_4 is in $\mathcal{Q}_4^{top} - \mathcal{F}$ and is such that $r_4v \neq g_2$. Let $(r_4v, \eta; g_2)$ be a Hamiltonian path in $\mathcal{Q}_4^{bot} - \{r_3\}$. The path $(g_1, \xi v\eta; g_2)$ is the desired path of length ten in $\mathcal{Q}_4 - \mathcal{F}$. □

Lemma 4.2. *Let $\mathcal{F} = \{r_1, r_2, r_3, r_4\} \subset \mathcal{Q}_4$, and g_1 and g_2 be two green vertices in $\mathcal{Q}_4 - \mathcal{F}$ such that $\mathcal{N}(g_1) \not\subset \mathcal{F}$ and $\mathcal{N}(g_2) \not\subset \mathcal{F}$. Then either there exists a path $(g_1, \eta; g_2)$ in $\mathcal{Q}_4 - \mathcal{F}$ with the maximal possible length eight or else, there is a partition of $\mathcal{Q}_4$ into plates such that $\mathcal{F} \subset \mathcal{V}(\mathcal{Q}_4^{top})$, $\{g_1, g_2\} \subset \mathcal{V}(\mathcal{Q}_4^{bot})$, and the maximal length of a path $(g_1, \eta; g_2)$ is six.*

Proof. We consider two cases: (A) there exists a partition such that $\mathcal{F} \subset \mathcal{V}(\mathcal{Q}_n^{top})$; and (B) every partition has deleted vertices in each plate.

Case A. Assume that $\mathcal{F} \subset \mathcal{V}(\mathcal{Q}_n^{top})$ for some partition. If g_1, g_2 are on the bottom plate, then by Corollary 3.12, there is a path $(g_1, \eta; g_2)$ of length $2^{n-1} - 2 = 6$ in the bottom plate. There is no path longer than that since no path from g_1 to g_2 can visit the top plate which has all the red vertices deleted. If both g_1 and g_2 are on the top plate, then there is a path $(g_1v, \xi; g_2v)$ of length six in the bottom plate. The path $(g_1, v\xi v; g_2)$ is the desired path of length eight in $\mathcal{Q}_4 - \mathcal{F}$. If g_1, g_2 are not on the same plate we can assume without loss of generality that g_1 is on the top plate and g_2 is on the bottom plate. There is a path $(g_1v, \xi; g_2)$ that is Hamiltonian for the bottom plate. Then the path $(g_1, v\xi; g_2)$ is the desired path of length eight in $\mathcal{Q}_4^{top} - \mathcal{F}$.

Case B. Split $\mathcal{Q}_4$ into two plates such that $g_1 \in \mathcal{V}(\mathcal{Q}_4^{top})$ and $g_2 \in \mathcal{V}(\mathcal{Q}_4^{bot})$. Then we have to consider the following three subcases: (1) $|\mathcal{F}^{top}| = 4$; (2) $|\mathcal{F}^{top}| = 3$; and (3) $|\mathcal{F}^{top}| = 2$ for all possible splittings that separate g_1 from g_2.

Subcase (1). If $(g_1v, \xi; g_2)$ is a Hamiltonian path in $\mathcal{Q}_4^{bot}$ then $(g_1, v\xi; g_2)$ is the desired path of length eight in $\mathcal{Q}_4 - \mathcal{F}$.

Subcase (2). We assume that r_4 is the deleted vertex in the bottom plate. Up to isomorphism the possible configurations for $\mathcal{F}^{top}$ are (a) $\mathcal{F}^{top} = \{g_1x, g_1y, g_1z\}$; and (b) $\mathcal{F}^{top} = \{g_1x, g_1y, g_1xyz\}$, where $x, y, z \in \mathbf{S}$.

(a) In this case, up to isomorphism, $\mathcal{F}^{bot} = \{g_1vxy\}$ is the only possibility for $\mathcal{F}^{bot}$. Again, up to isomorphism, there are three possibilities: $g_2 = g_1vxyz$, $g_2 = g_1vx$, or $g_2 = g_1vz$. In the first case the path $(g_1, vxzvyxvx; g_2)$ does the job; in the second case the path $(g_1, vyzvxyvz; g_2)$ does the job; and in the last case $(g_1, vxzyvxvy; g_2)$ does the job.

(b) Observe that there is a path $(g_1, \xi; g_3)$ of length two in $\mathcal{Q}_4^{top} - \mathcal{F}$ such that $g_3v \neq r_4$. There is also a path $(g_3v, \eta; g_2)$ of length five in $\mathcal{Q}_4^{bot} - \{r_4\}$ (Theorem 3.1). The path $(g_1, \xi v\eta; g_2)$ is the desired path of length eight in $\mathcal{Q}_4 - \mathcal{F}$.

Subcase (3). For this subcase there is a red vertex r and letters $x, y \in \mathbf{S}$ such that $\mathcal{F} = \{r, rxy, rvz, rxyvz\}$. Then, up to isomorphism, there are two possible values for $g_1 : g_1 = rx$; or $g_1 = rz$. The following table shows fault-free paths of length eight from either of the two values of g_1 to every green vertex g_2 in the bottom plate.

g_2	path from $g_1 = rx$	path from $g_1 = rz$
rv	$vzvyxzvy$	$xyxvzxyx$
$rxyv$	$vzvyxzvx$	$xyxvzyxy$
$rxzv$	$zyxvzxyz$	$xyxvzxyz$
$ryzv$	$vzvyxzvz$	$yxyvzyxz$

□

For the proof of the following general result we use the previous two lemmas.

Theorem 4.3. *Let $n \geq 5$ and f be integers with $1 \leq f \leq n$. Let also $\mathcal{F}$ be a set of red vertices in $\mathcal{Q}_n$ with cardinality f. Then for any pair of green vertices g_1, g_2 in $\mathcal{Q}_n - \mathcal{F}$ such that $\mathcal{N}(g_1) \not\subset \mathcal{F}$ and $\mathcal{N}(g_2) \not\subset \mathcal{F}$, there exists a path $(g_1, \eta; g_2)$ with the maximal possible length $2^n - 2f$ in $\mathcal{Q}_n - \mathcal{F}$.*

Proof. First, let us observe that if $f = 1$ the statement of the theorem follows immediately from $[1, 1, 0, 1] = 2$ ([LW], [CG2, Lemma 3.5]), and that if $f = 2$, the statement of the theorem follows from $[3, 1, 0, 1] = 4$ ([CG2, Lemma 3.17]).

Now, let $3 \leq f \leq n$ and $n \geq 5$. The proof is by induction on n. Without loss of generality, we can assume that for any given partition of $\mathcal{Q}_n$ into two plates $\mathcal{Q}_n^{top}$ and $\mathcal{Q}_n^{bot}$ we have $f_0 = |\mathcal{F}^{bot}| \leq |\mathcal{F}^{top}| = f_1$. The following two cases exhaust all possible situations: (1) for some partition we have $1 \leq f_0 \leq f_1 \leq n - 2$; and (2) for every partition $f_1 \geq n - 1$.

Case (1). Split $\mathcal{Q}_n$ into two plates such that $1 \leq f_0 \leq f_1 \leq n - 2$. Then we have to consider three subcases: (a) g_1 and g_2 are on the top plate; (b) g_1 and g_2 are on the bottom plate; and (c) g_1 is on the top plate and g_2 is on the bottom plate.

Subcase (a). By the induction hypothesis (if $n > 5$), by Lemma 4.1 (if $n = 5$ and $f_1 = 3$), or by $[3, 1, 0, 1] = 4$ (if $n = 5$ and $f_1 = 2$) there exists a path $(g_1, \xi; g_2)$ of length $2^{n-1} - 2f_1$ in $\mathcal{Q}_n^{top} - \mathcal{F}$. Since the path is long enough ($2^{n-1} - 2f_1 > 2f_0$ if $n \geq 5$), there are words μ, ν such that $\xi = \mu\nu$, and neither $g_1\mu' v$ nor $g_1\mu v$ is in $\mathcal{F}$. By Theorem 3.1, there is a path $(g_1\mu' v, \eta; g_1\mu v)$ of length at least $2^{n-1} - 2f_0 - 1$ in $\mathcal{Q}_n^{bot} - \mathcal{F}$. Then $(g_1, \mu' v\eta v\nu; g_2)$ is a path of length at least $2^n - 2f$ in $\mathcal{Q}_n - \mathcal{F}$. Since that is the maximal possible length of a path that connects g_1 to g_2 in $\mathcal{Q}_n - \mathcal{F}$, the proof of this subcase is completed.

Subcase (b). By the induction hypothesis (if $n > 5$), by $[3, 1, 0, 1] = 4$ (if $n = 5$ and $f_0 = 2$), or by $[1, 1, 0, 1] = 2$ (if $n = 5$ and $f_0 = 1$) there exists a path $(g_1, \xi; g_2)$ of length $2^{n-1} - 2f_0$ in $\mathcal{Q}_n^{bot} - \mathcal{F}$. Since the path is long enough ($2^{n-1} - 2f_0 > 2f_1 + 2$ if $n \geq 5$), there are words μ, ν such that $\xi = \mu\nu$, neither $g_1\mu' v$ nor $g_1\mu v$ is in $\mathcal{F}$, $\mathcal{N}(g_1\mu' v)^{top} \not\subset \mathcal{F} \cup \{g_1\mu v\}$, and $\mathcal{N}(g_1\mu v)^{top} \not\subset \mathcal{F} \cup \{g_1\mu' v\}$. By Theorem 3.1 (if $f_1 \leq n - 3$), or by Theorem 3.9 (if $f_1 = n - 2$), there is a path $(g_1\mu' v, \eta; g_1\mu v)$ of length at least $2^{n-1} - 2f_1 - 1$ in $\mathcal{Q}_n^{top} - \mathcal{F}$. Then $(g_1, \mu' v\eta v\nu; g_2)$ is a path of length $2^n - 2f$ in $\mathcal{Q}_n - \mathcal{F}$. Since that is the maximal possible length of a path that connects g_1 to g_2 in $\mathcal{Q}_n - \mathcal{F}$, the proof of this subcase is completed.

Subcase (c). Since $n \geq 5$, there exists a red vertex r in $\mathcal{Q}_n^{top}$ such that r is not in $\mathcal{F}$, $\mathcal{N}(g_1)^{top} \not\subset \mathcal{F} \cup \{r\}$, and $rv \neq g_2$. By Theorem 3.1 (if $f_1 \leq n - 3$), or by Theorem 3.9 (if $f_1 = n - 2$), there is a path $(g_1, \xi; r)$ of length at least $2^{n-1} - 2f_1 - 1$ in $\mathcal{Q}_n^{top} - \mathcal{F}$. By the induction hypothesis (if $n > 5$), by $[3, 1, 0, 1] = 4$ (if $n = 5$ and $f_0 = 2$), or by $[1, 1, 0, 1] = 2$ (if $n = 5$ and $f_0 = 1$) there is a path $(rv, \eta; g_2)$ of length

$2^{n-1} - 2f_0$ in $\mathcal{Q}_n^{bot} - \mathcal{F}$. Then $(g_1, \xi v \eta; g_2)$ is a path of length at least $2^n - 2f$ in $\mathcal{Q}_n - \mathcal{F}$. Since that is the maximal possible length of a path that connects g_1 to g_2 in $\mathcal{Q}_n - \mathcal{F}$, the proof of this subcase is completed.

Case (2). This case can occur only if $f = n$. Since all deleted vertices are red, it follows from Proposition 3.8 that $\mathcal{F} = \mathcal{N}(g)$ for some green vertex g in the top plate. By hypothesis g has to be different from g_1 and g_2. Without loss of generality, we can assume that g_1 is on the top plate and g_2 is on the bottom plate. Since $n \geq 5$, there is a red vertex r in $\mathcal{Q}_n^{top} - \mathcal{F}$ such that $rv \neq g_2$. By Theorem 3.11, there is a path $(g_1, \xi; r)$ of length at least $2^{n-1} - 2f_1 - 1$ in $\mathcal{Q}_n^{top} - \mathcal{F}$. Also, since $[1,1,0,1] = 2$, there is a path $(rv, \eta; g_2)$ of length $2^{n-1} - 2$ in $\mathcal{Q}_n^{bot} - \mathcal{F}$. Then $(g_1, \xi v \eta; g_2)$ is a path of length at least $2^n - 2f$ in $\mathcal{Q}_n - \mathcal{F}$. Since that is the maximal possible length of a path that connects g_1 to g_2 in $\mathcal{Q}_n - \mathcal{F}$, the proof of this case is completed. □

5. The case $n = 5$

In this section we prove the case $n = 5$ of the main theorem. Notice that our theorem improves by 2 (in the case $n = 5$) the estimate provided by Fu's theorem and as the example given in the introduction shows, 8 is the exact upper bound for f in this case.

Theorem 5.1. *Let $n = 5$ and f be an integer such that $0 \leq f \leq 3n - 7 = 8$. Then for any set of vertices $\mathcal{F}$ in $\mathcal{Q}_n$ of cardinality f there exists a cycle in $\mathcal{Q}_n - \mathcal{F}$ of length at least $2^n - 2f$.*

Proof. Since Fu's Theorem covers the cases $0 \leq f \leq 6$, we just need to prove our claim when $f = 7$ and $f = 8$. Without loss of generality, we assume that for any splitting of the hypercube we have $0 \leq f_0 = |\mathcal{F}^{bot}| \leq |\mathcal{F}^{top}| = f_1 \leq 8$. Let $(k, 8-k)$ mean that $f_0 = k, f_1 = 8 - k$.

Part One. For this part $f = 8$.

The strategy of the proof is as follows: We consider all possible cases in the following order: $(0,8), (1,7), (4,4), (3,5)$ leaving for the end the case when each splitting is of type $(2,6)$.

The case $(0,8)$ is solved by any Hamiltonian cycle in the bottom plate.

For the case $(1,7)$ let us assume, without loss of generality, that the faulty vertex on the bottom plate is a red vertex r. Then there exists a red vertex r_1 in $\mathcal{Q}_5^{top} - \mathcal{F}$ and a letter $x \in \mathbf{S}$ different from v such that neither $r_1 v$ nor $r_1 x v$ is in $\mathcal{F}$. Let $(r_1 v, \xi; r_1 x v)$ be a path of length $2^{n-1} - 2 - 1 = 13$ in $\mathcal{Q}_5^{bot} - \mathcal{F}$ (Theorem 3.1). Then $(r_1, v \xi v x)$ is the desired cycle of length $2^n - 2f = 16$ in $\mathcal{Q}_5 - \mathcal{F}$.

For the case $(4,4)$ we consider three subcases: (1) each plate has faulty vertices of both colors; (2) all faulty vertices of $\mathcal{Q}_5^{top}$ are of the same color and are not contained in any three dimensional subcube of $\mathcal{Q}_5^{top}$; (3) all faulty vertices of $\mathcal{Q}_5^{top}$ are of the same color and are contained in a three dimensional subcube of $\mathcal{Q}_5^{top}$.

Subcase (1). Since each plate contains at most three deleted vertices of a given color, there are two non-deleted vertices r, g in the bottom plate such that (a) rv and gv are not in $\mathcal{F}$; (b) $\mathcal{N}^{bot}(r) \not\subset \mathcal{F}^{bot} \cup \{g\}$; (c) $\mathcal{N}^{bot}(g) \not\subset \mathcal{F}^{bot} \cup \{r\}$; (d) $\mathcal{N}^{top}(rv) \not\subset \mathcal{F}^{top} \cup \{gv\}$; (e) $\mathcal{N}^{top}(gv) \not\subset \mathcal{F}^{top} \cup \{rv\}$. Then, by Theorem 3.11, there exists a path $(r, \xi; g)$ of length at least $2^{n-1} - 2f_0 - 1 = 7$ in $\mathcal{Q}_5^{bot} - \mathcal{F}$ and a path $(gv, \eta; rv)$ of length at least $2^{n-1} - 2f_1 - 1 = 7$ in $\mathcal{Q}_5^{top} - \mathcal{F}$. The cycle $(r, \xi v \eta v)$ is the desired cycle of length at least $2^n - 2f = 16$ in $\mathcal{Q}_5 - \mathcal{F}$.

Subcase (2). We can assume that all faulty vertices in the top plate are red. Let (r, ξ) be a cycle of length at least $2^{n-1} - 2f_0 = 8$ in $\mathcal{Q}_5^{bot} - \mathcal{F}$ (Theorem 1.1). Since this cycle has four distinct red vertices, we can assume that r, ξ have been selected in such a way that $\mathcal{N}(rv) \not\subset \mathcal{F}^{top}$ and $\mathcal{N}(r\xi''v) \not\subset \mathcal{F}^{top}$. Then, by Lemma 4.2, there is a path $(rv, \eta; r\xi''v)$ of length at least eight in $\mathcal{Q}_5^{top} - \mathcal{F}$. The cycle $(r, v\eta v(\xi'')^R)$ is the desired cycle of length at least $2^n - 2f = 16$ in $\mathcal{Q}_5 - \mathcal{F}$.

Subcase (3). We split $\mathcal{Q}_5^{top}$ into two plates $\mathcal{Q}_5^{top,top}$ and $\mathcal{Q}_5^{top,bot}$ such that all deleted vertices in $\mathcal{Q}_5^{top}$ are red and are contained in $\mathcal{Q}_5^{top,top}$. Let u be the letter such that $\mathcal{Q}_5^{top,bot} = u\mathcal{Q}_5^{top,top}$. We also denote $v\mathcal{Q}_5^{top,top}$ by $\mathcal{Q}_5^{bot,top}$ and $v\mathcal{Q}_5^{top,bot}$ by $\mathcal{Q}_5^{bot,bot}$.

If all deleted vertices of the bottom plate are in $\mathcal{Q}_5^{bot,bot}$ then we can proceed as follows. Let r_1, r_2 be any two red vertices in $\mathcal{Q}_5^{top,bot}$. Then, by Corollary 3.10, there exists a path $(r_1, \xi; r_2)$ of length $2^{n-2} - 2 = 6$ in $\mathcal{Q}_5^{top,bot}$ and a path $(r_2uv, \eta; r_1uv)$ of length $2^{n-2} - 2 = 6$ in $\mathcal{Q}_5^{bot,top}$. Then $(r_1, \xi uv\eta vu)$ is the desired cycle of length $2^n - 2f = 16$ in $\mathcal{Q}_5 - \mathcal{F}$.

If there are less than four deleted vertices in $\mathcal{Q}_5^{bot,bot}$ then there exist a red vertex r and a green vertex g in $\mathcal{Q}_5^{bot,bot} - \mathcal{F}$ such that $\mathcal{N}(r)^{bot} \not\subset \mathcal{F}^{bot} \cup \{g\}$ and $\mathcal{N}(g)^{bot} \not\subset \mathcal{F}^{bot} \cup \{r\}$. Then, by Theorem 3.11, there exists a path $(g, \eta; r)$ of length at least $2^{n-1} - 2f_0 - 1 = 7$ in $\mathcal{Q}_5^{bot} - \mathcal{F}$. Finally, let $(rv, \xi; gv)$ be a path of length 7 in $\mathcal{Q}_5^{top,bot}$. Then $(r, v\xi v\eta)$ is the desired cycle of length at least $2^n - 2f = 16$ in $\mathcal{Q}_5 - \mathcal{F}$.

In the case $(3,5)$, according to Theorem 1.1, there exists a cycle (a, η) of length at least eight in $\mathcal{Q}_5^{top} - (\mathcal{F} \setminus \{b\})$, where $b \in \mathcal{F}^{top}$ and a is chosen to be b if the cycle passes through b, or else, any vertex in the cycle.

There are two subcases: (1) all faulty vertices of the bottom plate are of the same color; (2) there are faulty vertices of both colors in the bottom plate.

Subcase (1). We can assume that all faulty vertices in the bottom plate are red.

If a is green, produce a path $(a\eta' v, \xi; a(\eta^R)' v)$ of length ten in $\mathcal{Q}_5^{bot} - \mathcal{F}$. Such path exists according to Lemma 4.1. Then $(a(\eta^R)', \eta'^{*} v\xi v)$ is a cycle of length at least $10+2+6=18$ in $\mathcal{Q}_5 - \mathcal{F}$.

If a is red, produce a path $(a\eta'' v, \xi; a(\eta^R)'' v)$ of length ten in $\mathcal{Q}_5^{bot} - \mathcal{F}$. Such path exists according to Lemma 4.1. Then $(a(\eta^R)'', \eta''^{**} v\xi v)$ is a cycle of length at least $10+2+4=16$ in $\mathcal{Q}_5 - \mathcal{F}$.

Subcase (2). Without loss of generality we can assume that there are two red vertices and one green vertex in $\mathcal{F}^{bot}$.

If a is a green vertex, then at least one of the pairs $(a\eta' v, a(\eta^R)''' v)$ or $(a\eta''' v, a(\eta^R)' v)$ of green vertices does not contain a deleted vertex. Since both cases are symmetrical we shall consider only the case when $(a\eta' v, a(\eta^R)''' v)$ does not contain the deleted green vertex. Produce a Hamiltonian path $(a\eta' v, \xi; a(\eta^R)''' v)$ of length twelve in $\mathcal{Q}_5^{bot} - \mathcal{F}$. Such path exists since $[3,1,0,1]=4$ ([CG2, Lemma 3.17]). Then $(a(\eta^R)''', \eta'^{***} v\xi v)$ is a cycle of length at least $12+2+4=18$ in $\mathcal{Q}_5 - \mathcal{F}$.

If a is a red vertex then there are at least three more red vertices r_1, r_2, and r_3 in (a, η). At most one of the three green vertices $r_1 v$, $r_2 v$, or $r_3 v$ is a deleted vertex. Without loss of generality, we can assume that $r_1 v$ and $r_2 v$ are non-deleted vertices. Let $(r_1, \zeta; r_2)$ be the path along (a, η) that does not contain a. Clearly, the length of this path is at least two. Produce a Hamiltonian path $(r_2 v, \xi; r_1 v)$ of length twelve in $\mathcal{Q}_5^{bot} - \mathcal{F}$. Such path exists since $[3,1,0,1]=4$ ([CG2, Lemma 3.17]). Then $(r_1, \zeta v\xi v)$ is a cycle of length at least $12+2+2=16$ in $\mathcal{Q}_5 - \mathcal{F}$.

Finally, we consider the case when each splitting is of type $(2,6)$. We split $\mathcal{Q}_5^{bot}$ into two subcubes $\mathcal{Q}_5^{bot,top}$ and $\mathcal{Q}_5^{bot,bot}$ such that each one of them contains exactly one deleted vertex. This splitting partitions the top plate into $\mathcal{Q}_5^{top,top}$ and $\mathcal{Q}_5^{top,bot}$. One of these subcubes must contain exactly 5 deleted vertices and the other, say $\mathcal{Q}_5^{top,bot}$, must contain only one deleted vertex. Let r be the deleted vertex in $\mathcal{Q}_5^{top,bot}$ and let g_1, g_2 be vertices of the opposite parity in $\mathcal{Q}_5^{top,bot}$ such that neither $g_1 v$ nor $g_2 v$ is deleted (as always, v is the letter such that $\mathcal{Q}_5^{top} = v\mathcal{Q}_5^{bot}$). Let $(g_1, \xi; g_2)$ be a Hamiltonian path in $\mathcal{Q}_5^{top,bot} - \{r\}$, and $(g_2 v, \eta; g_1 v)$ be a path of length at least $2^4 - 4 - 2$ in $\mathcal{Q}_5^{bot} - \mathcal{F}^{bot}$ guaranteed by Corollary

3.10. Then the desired cycle in $\mathcal{Q}_5 - \mathcal{F}$ is $(g_1, \xi v \eta v)$. Observe that in this case the cycle is actually of length $6 + 2 + 2^4 - 4 - 2 = 18$.

This finishes the proof for $f = 8$.

Part Two. For this part $f = 7$.

The strategy is to cover the cases $(0, 7), (1, 6)$, and $(3, 4)$ first so that the remaining case is when all the splittings are of type $(2, 5)$.

The case $(0, 7)$ is solved by any Hamiltonian cycle in the bottom plate.

For the case $(1, 6)$ let us assume, without loss of generality, that the faulty vertex on the bottom plate is a red vertex r. Then there exist two red vertices r_1, r_2 in $\mathcal{Q}_5^{top} - \mathcal{F}$ and letters $x, y \in \mathbf{S} \setminus \{v\}$ such that none of r_1x, r_1xv, r_2y, r_2yv is in $\mathcal{F}$. Let g be any green vertex in $\mathcal{Q}_5^{bot} - \{r_1v, r_2v\}$. Since $[2, 0, 2, 0] = 4$ ([CG2, Lemma 5.6]), there exists a 2-path covering $(r_1v, \xi; r_2yv), (r_2v, \eta; r_1xv)$ of $\mathcal{Q}_5^{bot} - \{r, g\}$. Then $(r_1, v\xi vyv\eta vx)$ is the desired cycle of length $12 + 2 + 1 + 2 + 1 = 18$ in $\mathcal{Q}_5 - \mathcal{F}$.

The case $(3, 4)$ can be handled in the following way. As explained in the next paragraph, one can find in $\mathcal{Q}_5^{bot}$ a green vertex g and a red vertex r such that (1) $\mathcal{N}(r)^{bot} \not\subset \mathcal{F} \cup \{g\}$; (2) $\mathcal{N}(g)^{bot} \not\subset \mathcal{F} \cup \{r\}$; (3) $\mathcal{N}(rv)^{top} \not\subset \mathcal{F} \cup \{gv\}$; and (4) $\mathcal{N}(gv)^{top} \not\subset \mathcal{F} \cup \{rv\}$. Then, by Theorem 3.9 and Theorem 3.11, there exists a path $(r, \xi; g)$ of length at least nine in $\mathcal{Q}_5^{bot} - \mathcal{F}$, and a path $(gv, \eta; rv)$ of length at least seven in $\mathcal{Q}_5^{top} - \mathcal{F}$. The desired cycle of length $2^n - 2f = 18$ in $\mathcal{Q}_5 - \mathcal{F}$ is $(r, \xi v \eta v)$.

Here we explain how to choose the vertices r and g with the properties (1) – (4) mentioned above. Let $\widetilde{\mathcal{F}} = v\mathcal{F}^{top} \cup \mathcal{F}^{bot}$. Clearly $|\widetilde{\mathcal{F}}| \leq 7$. Without loss of generality we may assume that $r(\widetilde{\mathcal{F}}) < g(\widetilde{\mathcal{F}})$. In particular, not all the vertices in $\mathcal{F}^{top}$ are green. Hence for any green vertex $g \in \mathcal{Q}_5^{bot} - \widetilde{\mathcal{F}}$ we have $\mathcal{N}(gv) \cap \mathcal{Q}_5^{top} \neq \mathcal{F}^{top}$. In the extreme case when all the vertices in $\widetilde{\mathcal{F}}$ are of the same color, which by assumption must be green, we choose g to be the only green vertex on the bottom plate that is not in $\widetilde{\mathcal{F}}$. Among the four red vertices of $\mathcal{Q}_5^{bot}$ that are at distance three from g, it is easy to find a vertex r to satisfy conditions (1) – (4). If not all vertices in $\widetilde{\mathcal{F}}$ are green, let g_1, g_2 be two distinct green vertices in $\mathcal{Q}_5^{bot} - \widetilde{\mathcal{F}}$. Let us declare a red vertex in $\mathcal{Q}_5^{bot}$ to be *bad* if it is in $\widetilde{\mathcal{F}}$ or if all of its neighbors in the bottom plate are in $\widetilde{\mathcal{F}}$. There are at most 5 bad red vertices in $\mathcal{Q}_5^{bot}$: at most 3 deleted and at most two whose all neighbors in $\mathcal{Q}_5^{bot}$ are in $\widetilde{\mathcal{F}}$. On the other hand $|(S_3(g_1) \cup S_3(g_2)) \cap \mathcal{Q}_5^{bot}| \geq 6$. Therefore there is a red vertex in $(S_3(g_1) \cup S_3(g_2)) \cap \mathcal{Q}_5^{bot}$ that is not bad and that one is our choice of r.

Without loss of generality we may assume that $r \in S_3(g_1)$. Then g_1 is our choice of g.

Finally when all the splittings are of type $(2,5)$ we can proceed exactly as we did in Part one when all the splittings were of type $(2,6)$ and as it was pointed out there we can produce a cycle of length 18 in $\mathcal{Q}_5 - \mathcal{F}$. □

6. Proof of the main theorem for $n > 5$

Now we are ready to prove our main theorem for every $n > 5$.

Theorem 6.1. *Let $n \geq 5$ and f be integers with $0 \leq f \leq 3n-7$. Then for any set $\mathcal{F}$ of vertices in $\mathcal{Q}_n$ of cardinality f there exists a cycle in $\mathcal{Q}_n - \mathcal{F}$ of length at least $2^n - 2f$.*

Proof. The proof is by induction. Thanks to Fu's theorem (Theorem 1.1) we can assume that $f > 2n-4$, thus $f > n+1$. Therefore, there exists a splitting of $\mathcal{Q}_n$ such that $2 \leq f_0 = |\mathcal{F}^{bot}| \leq f_1 = |\mathcal{F}^{top}| \leq 3n-9$.

The case $n = 5$ was considered in the previous section.

Now let $n \geq 6$. We have to consider two cases: (1) $f_0 = 2$, $f_1 = 3n-9$; and (2) $2 \leq f_0 \leq f_1 \leq 3n-10$.

Case (1). There are two possibilities: (i) all vertices in $\mathcal{F}^{top}$ are of the same color, and all vertices of $\mathcal{F}^{bot}$ are of the opposite color; or (ii) there exists a vertex in $\mathcal{F}^{top}$ with the same color as a vertex in $\mathcal{F}^{bot}$.

Subcase (i). Without loss of generality, we can assume that the vertices in $\mathcal{F}^{top}$ are green and the vertices in $\mathcal{F}^{bot}$ are red. Let g_1 be any vertex in $\mathcal{F}^{top}$. By the induction hypothesis, there exists a cycle (g,ξ) of length at least $2^{n-1} - 2(f_1 - 1)$ in $\mathcal{Q}_n^{top} - (\mathcal{F}^{top} \setminus \{g_1\})$, where g is chosen to be g_1, if g_1 is in the cycle, or g is any green vertex in the cycle, otherwise. Since $[3,1,0,1] = 4$ ([CG2, Lemma 3.17]), there is a path $(g\xi' v, \eta; g(\xi^R)' v)$ of length $2^{n-1} - 4$ in $\mathcal{Q}_n^{bot} - \mathcal{F}$. Then $(g\varphi(\xi), \xi'^* v\eta v)$ is a cycle of length $2^n - 2f + 2$ in $\mathcal{Q}_n - \mathcal{F}$.

Subcase (ii). Without loss of generality, we can assume that there is a green vertex g_1 in $\mathcal{F}^{top}$ and a green vertex g_2 in $\mathcal{F}^{bot}$. By the induction hypothesis, there exists a cycle (g,ξ) of length at least $2^{n-1} - 2(f_1 - 1)$ in $\mathcal{Q}_n^{top} - (\mathcal{F}^{top} \setminus \{g_1\})$, where g is chosen to be g_1, if g_1 is in the cycle, or g is any green vertex in the cycle, otherwise. We can also assume, without loss of generality, that $g\xi' v \neq g_2$.

If the second vertex in $\mathcal{F}^{bot}$ is also green, then $g\xi'' v$ and $g(\xi^R)'' v$ are two red vertices in $\mathcal{Q}_n^{bot} - \mathcal{F}$. Therefore, since $[3,1,0,1] = 4$ ([CG2, Lemma 3.17]), there exists a path $(g\xi'' v, \eta; g(\xi^R)'' v)$ of length $2^{n-1} - 4$

in $\mathcal{Q}^{bot}-\mathcal{F}$. Then $(g(\xi^R)'', (\xi^{**})''v\eta v)$ is a cycle of length at least 2^n-2f in $\mathcal{Q}_n-\mathcal{F}$.

If the second vertex in $\mathcal{F}^{bot}$ is red, then either $g(\xi^R)'v$ or $g(\xi^R)'''v$ is not in $\mathcal{F}^{bot}$. In the first case, since $[3,1,0,1]=4$, there exists a path $(g\xi'v, \eta; g(\xi^R)'v)$ of length $2^{n-1}-4$ in $\mathcal{Q}_n^{bot}-\mathcal{F}$. Then $(g(\xi^R)', (\xi^*)'v\eta v)$ is a cycle of length at least 2^n-2f+2 in $\mathcal{Q}_n-\mathcal{F}$. In the second case, again according to $[3,1,0,1]=4$, there exists a path $(g\xi'v, \eta; g(\xi^R)'''v)$ of length $2^{n-1}-4$ in $\mathcal{Q}_n^{bot}-\mathcal{F}$. Then $(g(\xi^R)''', (\xi^{***})'v\eta v)$ is a cycle of length at least 2^n-2f in $\mathcal{Q}_n-\mathcal{F}$.

Case (2). Observe that $3n-10=3(n-1)-7$. Therefore, by the induction hypothesis, there exists a cycle (r,ξ) of length at least $2^{n-1}-2f_1$ in $\mathcal{Q}_n^{top}-\mathcal{F}$, where r is some vertex in $\mathcal{Q}_n^{top}-\mathcal{F}$. The cycle (r,ξ) has length $2^{n-1}-2f_1\geq 4f_0$ with equality if and only if $n=6, f_0=5, f_1=6$.

If at least one out of every four consecutive vertices along (r,ξ) is adjacent to a faulty vertex in $\mathcal{Q}_4^{bot}$, then $n=6, f_0=5, f_1=6$ and all the vertices of $\mathcal{Q}_6^{top}$ that are adjacent to the five faulty vertices of $\mathcal{Q}_6^{bot}$ divide the cycle (r,ξ) into five paths each of length 4. Let a,b be two vertices in the cycle (r,ξ) at distance two from each other and such that $av, bv\notin\mathcal{F}$, $\mathcal{N}(av)\neq\mathcal{F}^{bot}$ and $\mathcal{N}(bv)\neq\mathcal{F}^{bot}$. Observe that the parity of all the faulty vertices in the bottom plate is opposite to the parity of av and bv. Let η, x, y be a word and two letters such that $(r,\xi)=(a,\eta xy)$, with $a\eta=b$. By Theorem 4.3 there exists a path $(bv,\nu;av)$ in $\mathcal{Q}_6^{bot}-\mathcal{F}^{bot}$ of length $2^{n-1}-2f_0$. The desired cycle in $\mathcal{Q}_6-\mathcal{F}$ is $(a,\eta v\nu v)$.

We can assume now that along the cycle (r,ξ) there are four consecutive vertices adjacent to non-faulty vertices in $\mathcal{Q}_6^{bot}$. Thus there are words η, θ, and a letter $w\in\mathbf{S}$ such that $\xi=\eta w\theta$, and none of $r\eta'v$, $r\eta v$, $r\eta wv$, or $r\eta w\varphi(\theta)v$ is in $\mathcal{F}$. Split $\mathcal{Q}_n^{bot}$ further into two plates $\mathcal{Q}', \mathcal{Q}''$ such that $\mathcal{Q}'=w\mathcal{Q}''$. Without loss of generality, we can assume that $f_0'=|\mathcal{F}\cap\mathcal{V}(\mathcal{Q}')|\leq f_0''=|\mathcal{F}\cap\mathcal{V}(\mathcal{Q}'')|$, and that $r\eta'v$ and $r\eta v$ are in $\mathcal{Q}'$. We shall consider separately the case when $f_0'=0$ and the case when $f_0'\geq 1$.

First, assume that $f_0'=0$ and let $a\in\mathcal{F}^{bot}$. It is clear that if $n\geq 7$ then $f_0''-1\leq\frac{3n-7}{2}-1\leq 3n-13=3(n-2)-7$ and if $n=6$ then $f_0''-1\leq 5-1=4=2(n-2)-4$. Therefore, by Fu's theorem (if $n=6$), or by the induction hypothesis (if $n\geq 7$), there exists a cycle (b,ζ) of length at least $2^{n-2}-2(f_0''-1)$ in $\mathcal{Q}''-(\mathcal{F}\setminus\{a\})$, where b is some vertex in $\mathcal{Q}''-\mathcal{F}$. There are two subcases to consider: (i) a is in the cycle (b,ζ); and (ii) a is not in the cycle (b,ζ).

Subcase (i). a is in the cycle (b, ζ). Then there exist words μ and ν such that $\zeta = \mu\nu$ and $a = b\mu$. There are four possibilities.

(a) $\{b\mu''w, a\varphi(\nu)w\} \cap \{r\eta'v, r\eta v\} = \emptyset$.

Let $(r\eta'v, \alpha; a\varphi(\nu)w)$, $(b\mu''w, \beta; r\eta v)$ be a 2−path covering of $\mathcal{Q}'$. Such path covering exists since $[0, 0, 2, 0] = 2$ ([CG2, Lemma 3.3]) and $[0, 0, 0, 2] = 4$ ([CG2, Lemma 3.13]). The desired cycle of length at least $2^n - 2f$ in $\mathcal{Q}_n - \mathcal{F}$ is $(r, \eta'v\alpha w\nu^*\mu''w\beta v w\theta)$.

(b) $\{b\mu''w, a\varphi(\nu)w\} = \{r\eta'v, r\eta v\}$.

Then $\{b\mu'w, a\varphi(\nu^*)w\} \cap \{r\eta'v, r\eta v\} = \emptyset$ and therefore the desired cycle could be constructed as in (a).

(c) $b\mu''w = r\eta v$ and $a\varphi(\nu)w \neq r\eta'v$.

Let $(r\eta'v, \alpha; a\varphi(\nu)w)$, be a Hamiltonian path for $\mathcal{Q}' - \{r\eta v\}$. Such Hamiltonian path exists since $[1, 1, 0, 1] = 2$. The desired cycle of length at least $2^n - 2f$ in $\mathcal{Q}_n - \mathcal{F}$ is $(r, \eta'v\alpha w\nu^*\mu''wvw\theta)$.

(d) $b\mu''w \neq r\eta v$ and $a\varphi(\nu)w = r\eta'v$.

This case is similar to case (c).

Subcase (ii). a is not in the cycle (b, ζ). By replacing b, if necessary, with some other vertex in the cycle, we can assume that $\{bw, b\zeta'w\} \cap \{r\eta'v, r\eta v\} = \emptyset$. Let $(r\eta'v, \alpha; bw)$, $(b\zeta'w, \beta; r\eta v)$ be a 2−path covering of $\mathcal{Q}'$. Such path covering exists since $[0, 0, 2, 0] = 2$ (see [D] or [CG2, Lemma 3.3]) and $[0, 0, 0, 2] = 4$ ([CG2, Lemma 3.13]). The desired cycle of length at least $2^n - 2f$ in $\mathcal{Q}_n - \mathcal{F}$ is $(r, \eta'v\alpha w\zeta'w\beta v w\theta)$. (The cycle is actually of length at least $2^n - 2f + 2$.)

Now, let us assume that $f_0' \geq 1$. In this case if $n \geq 7$ then $f_0'' \leq \frac{3n-7}{2} - 1 \leq 3n - 13 = 3(n-2) - 7$ and if $n = 6$ then $f_0'' \leq 5 - 1 = 4 = 2(n-2) - 4$. Therefore, by Fu's theorem (if $n = 6$), or by the induction hypothesis (if $n \geq 7$), there exists a cycle (c, μ) of length at least $2^{n-2} - 2f_0''$ in $\mathcal{Q}_n'' - \mathcal{F}$. Since this cycle is long enough (or, more precisely, since $2^{n-2} - 2f_0'' > 2(f_0' + 2)$ when $n \geq 6$) there exist c and μ such that neither cw nor $c\mu'w$ is in $\mathcal{F} \cup \{r\eta'v, r\eta v\}$. Also, it follows from $f_0' \leq f_0''$ and $n \geq 6$ that $f_0' \leq \lfloor \frac{3n-7}{4} \rfloor \leq (n-2) - 2$. Therefore, Theorem 3.13 applies, and there is a path $(r\eta'v, \zeta; r\eta v)$ of length at least $2^{n-2} - 2f_0' - 1$ in $\mathcal{Q}' - \mathcal{F}$ that passes through the edge $\{cw, c\mu'w\}$. Without loss of generality, we can assume that $\zeta = \rho u \gamma$ for some words ρ, γ and a letter $u \in \mathbf{S}$, such that $r\eta'v\rho = cw$ and $r\eta v = c\mu'w\gamma$. The desired cycle of length at least $2^n - 2f$ in $\mathcal{Q}_n - \mathcal{F}$ is $(r, \eta'v\rho w\mu'w\gamma v w\theta)$. □

Appendix A. Path coverings with prescribed ends

Table 1 summarizes some of the results obtained in [CG2] on path coverings with prescribed ends in faulty hypercubes. The rows represent admissible combinations of M and C and the columns contain all the values of N and O such that $N+O \leq 3$. Each star in the table represents an impossible case. The missing entries in the table correspond to values of $[M,C,N,O]$ that we do not know yet. The inequalities in the table represent an upper or lower bound of the corresponding entry.

Table 1

$MC\backslash NO$	01	10	20	11	02	30	21	12	03
00	$\star$	1	2	$\star$	4	5	$\star$	≤ 6	$\star$
11	2	$\star$	$\star$	4	$\star$	$\star$	≤ 6	$\star$	≤ 6
20	$\star$	4	4	$\star$	5		$\star$		$\star$
22	$\star$	$\star$	$\star$	$\star$	4	$\star$	$\star$	≤ 6	$\star$
31	4	$\star$	$\star$	5	$\star$	$\star$		$\star$	
33	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	≤ 6
40	$\star$	5		$\star$			$\star$		$\star$
42	$\star$	$\star$	$\star$	$\star$	5	$\star$	$\star$		$\star$
44	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
51	5	$\star$	$\star$	≥ 5	$\star$	$\star$		$\star$	

Appendix B. Paths in $\mathcal{Q}_3$

Table 2 shows the two different paths that connect a red vertex r to a green vertex g in $\mathcal{Q}_3 - r_1$, where r_1 is a red vertex. We assume without loss of generality that $r = r_1xy$ for two different letters x, y. Let also z be a third letter different from x and y. The table gives the three essentially distinct cases of g. Each path is represented by the word that should be followed in order to go from r to g.

Table 2

g	First Path	Second Path
ry	$zxyxz$	$xzyxz$
rz	$yzxyx$	$xzyxy$
$rxyz$	$yzyxy$	$xzxyx$

The reader can observe that for each case the two paths together cover at least 7 different edges. Observe also that for each value of g the second vertex of either path (counted from r) is not contained in the other path. For example, when $g = ry$, the second vertex of the first path is rz and is not contained in the second path. At the same time the second vertex of the second path is rx and it is not contained in the first path.

Appendix C. Special cases for Theorem 3.11

Table 3 contains, up to isomorphisms, all the possible configurations of faults of mass 4 in $\mathcal{Q}_4$ with the property that each splitting separates the fault with exactly two vertices on each plate. For each type of fault there is also a complete, up to isomorphisms, list of possible distributions of the red and green terminals on the top plate. Finally the table shows paths of length at least 7 in $\mathcal{Q}_4 - \mathcal{F}$ for each case.

Table 3

$\mathcal{F}$	r	g	path
$\varepsilon, xy, zv, zvxy$	x	xz	$vxyzvxy$
	x	yz	$zvzyxzv$
	z	xz	$yzvxyzv$
$\varepsilon, xy, xzv, zvy$	x	xz	$vyxyzvyxy$
	x	yz	$zyvzyxyvz$
	z	xz	$yzvyxyzvy$
$\varepsilon, x, yzv, xyzv$	y	yx	$zxyvzxyxv$
	y	yz	$xvxyxzvyx$
	y	xz	$zxzvxyxzv$
	z	zy	$xyzvyzxzyvz$
	xyz	zy	$zvyzxzyvz$
$\varepsilon, xyz, v, xyzv$	x	xy	$zvzyxzyvyzx$
	x	xz	$yxzyvyzxyzv$
	x	yz	$zvzyxzyvy$
	z	xz	$yzxyvyxzyxv$
	z	xy	$yzvzyxvzy$
$\varepsilon, xyz, xv, yzv$	x	xy	$zxyzvyzxyzv$
	x	xz	$yxzyvzyxzyv$
	x	yz	$zxvxyzxvz$
	z	xz	$yzvyzxyzvyz$
	z	xy	$yzvyzxyzv$
	y	xy	$zyxvxzyxv$
	y	xz	$zyvzyxvyz$
	y	yz	$vyzxyzvyzxy$

The notation is as follows. ε denotes a fixed vertex that we can think of as the vertex $(0,0,0,0)$. Any other vertex a is represented by a word that would label a path from ε to a. Finally, each path is represented by the word that represents the steps to follow starting from the first terminal.

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