FUNCTIONAL DEPENDENCE ON SMALL SETS OF INDICES

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Abstract. Let $f : Y \to Z$ with $Y \subseteq X_I := \Pi_{i\in I} X_i$. Then

(a) $J \subseteq I$ is *essential* if there are $x, y \in Y$ such that $d(x, y) = J$ and $f(x) \neq f(y)$, where $d(x, y) := \{i \in I : x_i \neq y_i\}$; $J_f := \{i : \{i\}$ is essential$\}$; an essential J is *optimally essential* if no essential $J' \subseteq J$ satisfies $|J'| < |J|$; $\mathcal{J} \in \mathfrak{J}_f$ if $\mathcal{J}$ is a maximal family of pairwise disjoint optimally essential sets; $\lambda_f := \sup\{|\mathcal{J}| : \mathcal{J} \in \mathfrak{J}_f\}$.

(b) f *depends on* $J \subseteq I$ if $[x, y \in Y, x_J = y_J] \Rightarrow f(x) = f(y)$; $\mathcal{D}_f := \{J \subseteq I : f$ depends on $J\}$; $\mu_f := \min\{|J| : J \in \mathcal{D}_f\}$.

Theorem 1. $J \in \mathcal{D}_f \Rightarrow \lambda_f \leq |J|$.

Theorem 2. $\mathcal{J} \in \mathfrak{J}_f \Rightarrow \bigcup \mathcal{J} \in \mathcal{D}_f$.

That context is strictly set-theoretic. Henceforth let X_I and Z be spaces with Z Hausdorff, and let $f \in C(Y, Z)$. This is known: (*) if Y contains a σ-product then $J \in \mathcal{D}_f$ iff $J_f \subseteq J$. The authors give examples to show: $J_f \in \mathcal{D}_f$ in (*) can fail, if any one of the three hypotheses are omitted; $\mathcal{J}, \mathcal{J}' \in \mathfrak{J}_f$, with $|\mathcal{J}| \neq |\mathcal{J}'|$, can occur; $J \in \mathcal{D}_f \Rightarrow |J| > \lambda_f$ can occur; $\mathcal{J} \in \mathfrak{J}_f \Rightarrow |\mathcal{J}| < \lambda_f$ can occur; $|J| \leq \lambda_f \Rightarrow J \notin \mathcal{D}_f$ (hence, $\lambda_f < \mu_f$) can occur.

The authors' interest in (*) is motivated by their observation that when X_I has the κ-box topology, its obvious analogue, say $(*)_\kappa$, can fail. They propose and prove (what seem to be) appropriate modifications of $(*)_\kappa$.

1. Introduction and Historical Perspective

Conventions, Notation, Definitions 1.1. (a) Topological spaces considered here are not subjected to any special standing separation properties. Additional hypotheses are imposed as required.

(b) ω is the least infinite cardinal, $\mathfrak{c}$ is the cardinality of $[0, 1]$, and α and κ are infinite cardinals. For I a set and β an arbitrary cardinal

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we write $[I]^\beta := \{J \subseteq I : |J| = \beta\}$; the notation $[I]^{<\beta}$ is defined analogously. Also, for A and B sets, we write $A\Delta B := (A\backslash B)\cup(B\backslash A)$.

(c) For a set $\{X_i : i \in I\}$ of sets and $J \subseteq I$, we write $X_J := \Pi_{i\in J} X_i$; and for every *generalized rectangle* $A = \Pi_{i\in I} A_i \subseteq X_I$ the *restriction set of* A, denoted $R(A)$, is the set $R(A) = \{i \in I : A_i \neq X_i\}$. When each $X_i = (X_i, \mathcal{T}_i)$ is a space, the symbol $(X_I)_\kappa$ denotes X_I with the *κ-box topology*; this is the topology for which $\{\Pi_{i\in I} U_i : U_i \in \mathcal{T}_i, |R(U)| < \kappa\}$ is a base. Thus the ω-box topology on X_I is the usual product topology. We note that even when κ is regular, the intersection of fewer than κ-many sets, each open in $(X_I)_\kappa$, may fail to be open in $(X_I)_\kappa$.

(d) For $x, y \in X_I = \Pi_{i\in I} X_i$, the symbol $d(x, y)$ denotes the *difference set* of $\{x, y\}$—that is, the set $d(x, y) = \{i \in I : x_i \neq y_i\}$.

(e) For $p \in X_I = \Pi_{i\in I} X_i$, the *$\kappa$-$\Sigma$-product of X_I based at p* is the set $\Sigma_\kappa(p) := \{x \in X_I : |d(p, x)| < \kappa\}$.

(f) A set $Y \subseteq X_I$ is *κ-invariant* [12] provided for every $x, y \in Y$ and $J \in [I]^{<\kappa}$ the point $z \in X_I$ defined by $x_J = z_J$, $y_{I\backslash J} = z_{I\backslash J}$, is in Y.

(g) For spaces Y and Z we denote by $C(Y, Z)$ the set of continuous functions from Y into Z.

(h) If κ is a cardinal number then $D(\kappa)$ denotes the discrete space with cardinality κ.

Remarks 1.2. (a) In the notation of 1.1(e), the usual Σ-product based at $p \in X_I$ is the set $\Sigma(p) = \Sigma_{\omega^+}(p)$, and the "little σ-product" [7] is the set $\sigma(p) := \Sigma_\omega(p) \subseteq X_I$. If $|X_i| \geq 2$ for all $i \in I$ then $\pi_J[\Sigma_\kappa(p)] = X_J$ iff $J \in [I]^{<\kappa}$, so if the sets X_i are topological spaces then each $\Sigma_\kappa(p) \subseteq X_I$ is dense in $(X_I)_\kappa$.

(b) Clearly, every Σ_κ-space in X_I, also every generalized rectangle in X_I, is κ-invariant. (Note that the notion κ-invariant is closely related to, but different from, the notion of a subspace *invariant under projection* defined in [15].)

The following definition and lemma are strictly set-theoretic, in the sense that topology plays no role in their statements. For the applications, of course, X_i and Z will be spaces, and $f \in C(Y, Z)$.

Definition 1.3. *Let $f : Y \to Z$, with $Y \subseteq X_I = \Pi_{i\in I} X_i$.*

(a) *If $J \subseteq I$, then f depends on J if $[x, y \in Y$ and $x_J = y_J] \Rightarrow f(x) = f(y)$. In this case, $f_J : \pi_J[Y] \to Z$ is well-defined by the rule $f = f_J \circ \pi_J|_Y$;*

(b) *$J_f := \{i : i \in I$ and there exist $x, y \in Y$ such that $d(x, y) = \{i\}$ and $f(x) \neq f(y)\}$;*

(c) *$\mathcal{D}_f := \{J \subseteq I : f$ depends on $J\}$;*

(d) *$\mu_f := \min\{|J| : J \in \mathcal{D}_f\}$.*

Remark 1.4. The function f of Definition 1.3 depends on $J = \emptyset$ if and only if f is a constant function; in this case, the function $f_\emptyset = f_J$ is not defined. In what follows we use the notation f_J only with the understanding that $J \neq \emptyset$.

Lemma 1.5. *Let $Y \subseteq X_I$ and $f : Y \to Z$. If $J \in \mathcal{D}_f$ then $J_f \subseteq J$.*

Proof. For $i \in J_f$ there are $x, y \in Y$ with $d(x, y) = \{i\}$ and $f(x) \neq f(y)$. If $i \in J$ fails then $x_J = y_J$ with $f(x) \neq f(y)$, a contradiction. □

In Statement 1.6 below we use our notation and terminology to restate (the contrapositive of) Proposition 3.3 in [16], which appears in [16] without proof.

Statement 1.6. *Let $f : X_I \to Z$ be a function between the sets $X_I := \Pi_{i\in I} X_i$ and Z, and let $J_f \subseteq J \subseteq I$. Then*
(a) *$J \in \mathcal{D}_f$; and*
(b) *if X_I and Z are topological spaces and f is continuous, then f_J is also continuous.*

Discussion 1.7. In our view, Statement 1.6 is seriously flawed. First, Remark 1.4 shows for constant functions f that 1.6(b) is meaningless (f_J being then undefined since $J = \emptyset$)—but the condition that f be non-constant is not given in [16]. More seriously, Example 1 in Discussion 1.10 shows that even when X_i and Z are topological spaces and $f \in C(X_I, Z)$ is non-constant, the relation $J_f \in \mathcal{D}_f$ of 1.6(a) may fail; in that example one has $J_f = \emptyset$ (or $J_f = \{\bar{i}\}$ where $\bar{i} \in I$), but for $J \subseteq I$ one has $J \in \mathcal{D}_f$ if and only if $|I\backslash J| < \omega$ (see also Example 2.1 below).

In remarks parallel to those of Miščenko [16], Hušek [11] pointed the way to the following proposition.

Proposition 1.8. *Let X_I be a product space, Y be a subspace of X_I, Z be a Hausdorff space, and $f \in C(Y, Z)$. If there is $p \in X_I$ such that $\sigma(p) \subseteq Y$, then for each $J \subseteq I$: $J \in \mathcal{D}_f$ iff $J \supseteq J_f$.*

Remark 1.9. Proposition 1.8 is not explicitly stated in [11] but it can be found implicitly (without proof) there on page 33. For a full and direct proof of Proposition 1.8 see Corollary 1.12 below, and for a generalization see Lemma 2.32. The condition in Proposition 1.8 that Z is a Hausdorff space is not mentioned in [11], but its applications in [11] are in that setting. In Discussion 1.10 below we indicate the necessity of some of the hypotheses in (our formulation of) Proposition 1.8.

For Hušek's comments on this matter, see pp. 33 and 36 of [11].

Discussion 1.10. We give three simple examples showing that the conclusion of Proposition 1.8 can fail if any one of the three hypotheses (a) Z is a Hausdorff space, (b) f is continuous, and (c) there is $p \in X_I$ such that $\sigma(p) \subseteq Y$, is omitted.

Let $X_I = \Pi_{i\in I} X_i$ with $|I| > \omega$ and with each X_i a Hausdorff space such that $|X_i| > 1$. We will arrange in each case that f is non-constant on Y and $J_f = \emptyset$, so f does not depend on J_f.

Example 1: (a) fails, (b) and (c) hold. Let $Z \subseteq X_I$ be maximal with respect to the property $p, q \in Z, p \neq q \Rightarrow \sigma(p) \cap \sigma(q) = \emptyset$. Then $X_I = \bigcup_{p\in Z} \sigma(p)$ and the map $f : Y := X_I \to Z$ given by $f(x) := p \in Z$ if $x \in \sigma(p)$ is well-defined from X_I onto Z, and continuous when Z is given the indiscrete ("concrete") topology. (Here $J_f = \emptyset$. To preserve the foregoing essential properties, but with $J_f \neq \emptyset$, fix $\overline{p} \in Z$ and $\overline{i} \in I$, choose $q \in \sigma(\overline{p})$ so that $d(\overline{p}, q) = \{\overline{i}\}$ and now define $f(x)$ as before when $x \notin \sigma(\overline{p})$. When $x \in \sigma(\overline{p})$ define $f(x) = q$ whenever $x_{\overline{i}} = q_{\overline{i}}$ and $f(x) = \overline{p}$ otherwise. Then $f : X_I \to Z \cup \{q\}$, (a) fails and (b) and (c) hold, and $J_f = \{\overline{i}\} \neq \emptyset$.)

Example 2: (b) fails, (a) and (c) hold. Take data as in Example 1, but with Z now given the usual topology inherited from $Y = X_I$. Then f is not continuous since for $p \in Z$ the set $f^{-1}(p)$, which is $\sigma(p)$, is proper and dense in Y, hence is not closed. Again (c) holds, just as in (a), since $Y = X_I = \bigcup_{p\in X_I} \sigma(p)$.

Example 3: (c) fails, (a) and (b) hold. Choose any $Y \subseteq X_I$ such that $|Y \cap \sigma(p)| = 1$ for each $p \in X_I$. (It is easy to see, arguing as in [3, 2.4], that Y may even be chosen dense in X_I.) Then $f := \mathrm{id} : Y \to Z := Y$ are as required.

[As an alternative to Example 3, take $X_I := \{0, 1\}^I$ with $|I| \geq \omega$, fix $p \in X_I$ and set $Y := \{x \in X_I : |\{i \in I : x_i \neq p_i\}| \text{ is even}\}$. Then Y is again dense in X_I, and $f := \mathrm{id} : Y \to Z = Y$ are as required.]

Lemma 1.11. *Let X_I be a product space, $p \in X_I$, and $\sigma(p) \subseteq Y \subseteq X_I$. Let Z be a Hausdorff space, $f \in C(Y, Z)$, and $g := f|\sigma(p)$. Then*

(a) *$J_g = J_f$;*

(b) *$J_f \in \mathcal{D}_f$; and*

(c) *$J_g \in \mathcal{D}_g$.*

Proof. (a) We first show $J_g \subseteq J_f$. If $i \in J_g$ then there are $x, y \in \sigma(p) \subseteq Y$ such that $f(x) = g(x) \neq g(y) = f(y)$ and $d(x, y) = \{i\}$, so $i \in J_f$.

Now we show $J_f \subseteq J_g$. Given $i \in J_f$, there are $x, y \in Y$ such that $d(x, y) = \{i\}$ and $f(x) \neq f(y)$, and since Z is a Hausdorff space and $f \in C(Y, Z)$ there are basic neighborhoods U and V of x and y, respectively, such that $f[U] \cap f[V] = \emptyset$. Now define $x', y' \in X_I$ by

$x'_i = x_i$ and $y'_i = y_i$,
$x'_j = x_j = y_j = y'_j$ if $j \neq i$, $j \in R(U) \cup R(V)$,
$x'_j = y'_j = p_j$ if $j \in I \backslash (R(U) \cup R(V))$.

Then $x', y' \in \sigma(p)$ and $d(x', y') = \{i\}$, and from $x' \in U$, $y' \in V$ follows $g(x') = f(x') \neq f(y') = g(y')$, hence $i \in J_g$.

(b) Suppose there are $x, y \in Y$ such that $x_{J_f} = y_{J_f}$ and $f(x) \neq f(y)$. We claim there is a net $s(\lambda) = (x(\lambda), y(\lambda))$ in $\sigma(p) \times \sigma(p)$ such that

(i) $s(\lambda)$ converges to (x, y), and
(ii) $x(\lambda)_i = y(\lambda)_i$ for all λ and all $i \in J_f$.

Indeed, using the density of $\sigma(p) \times \sigma(p)$ in $Y \times Y$, begin with a net $s'(\lambda) = (x'(\lambda), y'(\lambda))$ in $\sigma(p) \times \sigma(p)$ which converges to (x, y) and define $x(\lambda)$ and $y(\lambda)$ by

$$\begin{aligned} x(\lambda) &:= \quad x'(\lambda) \quad , \text{ and} \\ y(\lambda)_i &:= \begin{cases} y'(\lambda)_i & \text{if} \quad i \notin J_f \\ x(\lambda)_i & \text{if} \quad i \in J_f \end{cases} \end{aligned} \tag{1}$$

The finite set $d(y'(\lambda), x'(\lambda))$ contains $d(y(\lambda), x(\lambda))$, so from $x'(\lambda) = x(\lambda) \in \sigma(p)$ follows $y(\lambda) \in \sigma(p)$; condition (i) holds since $x(\lambda)_i$ and $y(\lambda)_i$ converge to x_i and y_i, respectively, for all $i \in I$; and condition (ii) is obvious. The claim is established.

Since Z is a Hausdorff space and the nets $f(x(\lambda))$ and $f(y(\lambda))$ converge to $f(x)$ and $f(y)$, respectively, there is λ such that

$$g(x(\lambda)) = f(x(\lambda)) \neq f(y(\lambda)) = g(y(\lambda)). \tag{2}$$

We have $x(\lambda)_{J_f} = y(\lambda)_{J_f}$ by (1) and hence $x(\lambda)_{J_g} = y(\lambda)_{J_g}$ by (a), so $g(x(\lambda)) = g(y(\lambda))$, contrary to (2). It follows for $x, y \in Y$ that if $x_{J_f} = y_{J_f}$ then $f(x) = f(y)$—i.e., (b) holds.

(c) This is immediate from (a) and (b), since if $x, y \in \sigma(p)$ and $x_{J_f} = y_{J_f}$, then $g(x) = f(x) = f(y) = g(y)$. □

Corollary 1.12. *Let X_I, Y, Z, f and g be as in Lemma 1.11. Then for each $J \subseteq I$ these conditions are equivalent:*
(a) $J \in \mathcal{D}_f$;
(b) $J \in \mathcal{D}_g$;
(c) $J \supseteq J_f$; *and*
(d) $J \supseteq J_g$.

Proof. This is clear from Lemma 1.5 (where continuity of f is not assumed) and Lemma 1.11. □

Remark 1.13. We note that if p, Y, X_I, Z, f and g are as in Lemma 1.11, then there is a least (= smallest) set $J \subseteq I$ on which f depends, and a least set $J \subseteq I$ on which g depends. These sets coincide: we have $J = J_f = J_g$. Clearly in this case $\mu_f = \mu_g = |J|$.

2. Some more general results

We would like to generalize Proposition 1.8 (or, equivalently, Corollary 1.12) to the κ-box topology. One might expect that the appropriate legitimate generalization is this (this is the "obvious analogue" $(*)_\kappa$ of $(*)$ to which we refer in the Abstract):

(†) *Let X_I be a product space, Y be a subspace of $(X_I)_\kappa$, Z be a Hausdorff space, and $f \in C(Y, Z)$. If there is $p \in X_I$ such that $\Sigma_\kappa(p) \subseteq Y$, then $J_f \in \mathcal{D}_f$.*

The following example shows that statement (†) is invalid. In this example we have even $Y = (X_I)_\kappa$.

Example 2.1. *Let $\{X_i : i \in I\}$ be a family of discrete spaces and $\kappa > |I| \geq \omega$ be a cardinal number. Then $(X_I)_\kappa$ is discrete. As in Example 1 of Discussion 1.10, let $Z \subseteq X_I$ be maximal with respect to the property $[x, y \in Z, x \neq y] \Rightarrow \sigma(x) \cap \sigma(y) = \emptyset$. Then $X_I = \bigcup_{x \in Z} \sigma(x)$, and since $(X_I)_\kappa$ is discrete, the map $f : X_I \twoheadrightarrow Z$ given by $f(x) = p$ if $x \in \sigma(p)$ is continuous. Here $J_f = \emptyset$ so f, since it is not a constant function, does not depend on J_f. For a set $J \subseteq I$ we have $J \in \mathcal{D}_f$ if and only if $|I \setminus J| < \omega$.*

Example 2.1 shows that there are continuous functions f defined on subspaces of product spaces, equipped with the κ-box topology, such that $J_f \notin \mathcal{D}_f$. In seeking for a set that can replace J_f in a correct generalization to the κ-box topology of the cited works of Miščenko [16] and Hušek [11], we have been led to the concepts of essential and optimally essential sets. Before defining these concepts we introduce some terminology.

Terminology 2.2. (a) Given a set X and $\mathcal{F} \subseteq \mathcal{P}(X)$, a family $\mathcal{A} \subseteq \mathcal{F}$ is *$\mathcal{F}$-cellular*, or *cellular for $\mathcal{F}$*, if the elements of $\mathcal{A}$ are non-empty and pairwise disjoint. And $\mathbf{S}(\mathcal{F})$, the *Souslin number* of $\mathcal{F}$, is defined by $\mathbf{S}(\mathcal{F}) := \min\{\kappa :$ no $\mathcal{F}-$cellular family $\mathcal{A}$ satisfies $|\mathcal{A}| = \kappa\}$.

(b) The conventions of (a) are consistent with standard topological practice, where a *cellular family* for a topological space $X = (X, \mathcal{T})$ is (in our notation) a $\mathcal{T}$-cellular family, and $\mathbf{S}(X) = \mathbf{S}(X, \mathcal{T})$, the *Souslin number of* $X = (X, \mathcal{T})$, is (in our notation) $\mathbf{S}(\mathcal{T})$.

(c) We here follow the conventions mentioned in (a) and (b). That is, given a space $X = (X, \mathcal{T})$, we refer to a $\mathcal{T}$-cellular family simply as *cellular*, and we write $\mathbf{S}(X)$ in place of $\mathbf{S}(\mathcal{T})$.

Definition 2.3. Let $f : Y \to Z$, with $Y \subseteq X_I$.

(a) If $J \subseteq I$, then J is *essential* if there are $x, y \in Y$ such that $d(x, y) = J$ and $f(x) \neq f(y)$;

(b) J is *optimally essential* if J is essential and no essential set $J' \subseteq J$ satisfies $|J'| < |J|$;

(c) $\mathcal{O}_f := \{J \subseteq I : J \text{ is optimally essential}\}$;

(d) $\mathfrak{J}_f := \{\mathcal{J} : \mathcal{J} \text{ is maximal } \mathcal{O}_f\text{-cellular family}\}$;

(e) $\lambda_f := \sup\{|\mathcal{J}| : \mathcal{J} \in \mathfrak{J}_f\}$.

Evidently for data as in Definition 2.3 and $i \in I$, we have $i \in J_f$ if and only if $\{i\}$ is essential. The following proposition is then obvious.

Proposition 2.4. *Let $f : Y \to Z$ with $Y \subseteq X_I$, and fix $i \in I$. Then these conditions are equivalent.*

(a) *$i \in J_f$;*

(b) *$\{i\}$ is essential;*

(c) *$\{i\}$ is optimally essential—i.e., $\{i\} \in \mathcal{O}_f$.*

Proposition 2.5. *Let $f : Y \to Z$, with $Y \subseteq X_I$. If $J \subset I$ is essential, then there is an optimally essential set $J' \subseteq J$.*

Proof. Among all essential subsets of J, choose J' of minimal cardinality. Then $J' \in \mathcal{O}_f$. □

Proposition 2.6. *Let $f : Y \to Z$, with $Y \subseteq X_I$, and let $\mathcal{J} \in \mathfrak{J}_f$. Then $\{i\} \in \mathcal{J}$ for each $i \in J_f$. Hence $\lambda_f \geq |J_f|$.*

Proof. Let $i \in J_f$ and suppose that $\{i\} \notin \mathcal{J}$. If there is $J' \in \mathcal{J}$ such that $i \in J'$ then $\{i\} \neq J'$, hence we have $|\{i\}| < |J'|$. Since $\{i\} \in \mathcal{O}_f$ by Proposition 2.4(c), this contradicts the condition $J' \in \mathcal{O}_f$. Therefore there is no $J' \in \mathcal{J}$ such that $i \in J'$. The containment $\mathcal{J} \subseteq \mathcal{J} \cup \{i\}$ is then proper, with $\mathcal{J} \cup \{i\}$ $\mathcal{O}_f$-cellular, contrary to the maximality of $\mathcal{J}$. □

The following results indicate the utility of optimally essential sets.

Theorem 2.7. *Let $f : Y \to Z$, with $Y \subseteq X_I$, and let $J \subseteq I$. Then these conditions are equivalent.*

(a) *$J \in \mathcal{D}_f$;*

(b) *J intersects every essential set; and*

(c) *J intersects every optimally essential set.*

Proof. (a) ⇒ (b). Assume that $J \in \mathcal{D}_f$ and let J' be an essential set such that $J' \cap J = \emptyset$. Let $x, y \in Y$ be such that $d(x, y) = J'$ and $f(x) \neq f(y)$. Then $x_J = y_J$ and $f(x) = f(y)$, a contradiction.

(b) ⇒ (c). This is obvious.

(c) ⇒ (a). Suppose there are $x, y \in Y$ such that $x_J = y_J$ and $f(x) \neq f(y)$. Then $J' := d(x, y) \subset I \setminus J$ is an essential set. Let $J'' \subseteq J'$ be an optimally essential set, as given by Proposition 2.5. Clearly, $J \cap J'' = \emptyset$, which is a contradiction. □

Corollary 2.8. *Let $f : Y \to Z$, with $Y \subseteq X_I$, $J \subseteq I$, and $J \in \mathcal{D}_f$. If $J' := \{i \in J : i \in d(x, y)$ for some $x, y \in Y$, such that $f(x) \neq f(y)\}$ then $J' \in \mathcal{D}_f$.*

Proof. Suppose that $J' \notin \mathcal{D}_f$. Then there exist $x, y \in Y$ such that $x_{J'} = y_{J'}$ and $f(x) \neq f(y)$. Clearly $d(x, y)$ is essential, so $d(x, y) \cap J \neq \emptyset$ by Theorem 2.7(b). Thus $d(x, y) \cap J' \neq \emptyset$, which contradicts to $x_{J'} = y_{J'}$. □

The following theorem shows that if $\mathcal{J}$ is a maximal $\mathcal{O}_f$-cellular family then $\bigcup \mathcal{J} \in \mathcal{D}_f$ and therefore $\mu_f \geq \lambda_f$.

Theorem 2.9. *Let $f : Y \to Z$, with $Y \subseteq X_I$, and let $\mathcal{J} \in \mathfrak{J}_f$. Then*
(a) *$\bigcup \mathcal{J} \in \mathcal{D}_f$; and*
(b) *if $J \subseteq I$ is such that $J \in \mathcal{D}_f$ then $|J| \geq |\mathcal{J}|$.*

Proof. (a) Let $J := \bigcup \mathcal{J}$ and suppose there are $x, y \in Y$ such that $x_J = y_J$ and $f(x) \neq f(y)$. Then $J' := d(x, y) \subset I \setminus J$ is an essential set. Let $J'' \subseteq J'$ be an optimally essential set, as given by Proposition 2.5. Then $\mathcal{J} \cup \{J''\}$ is an $\mathcal{O}_f$-cellular family properly containing $\mathcal{J}$, contrary to the maximality of $\mathcal{J}$.
(b) Since $\mathcal{J}$ is $\mathcal{O}_f$-cellular, (b) is immediate from Theorem 2.7. □

Corollary 2.10. *Let $f : Y \to Z$, with $Y \subseteq X_I$. If $J \subseteq I$ is such that $J \in \mathcal{D}_f$ then $|J| \geq \lambda_f$, hence $\mu_f \geq \lambda_f$.*

Corollary 2.11. *Let $f : Y \to Z$, with $Y \subseteq X_I$, and let $\mathcal{J} \in \mathfrak{J}_f$. Then $|\bigcup \mathcal{J}| \geq \lambda_f$.*

Proof. $\bigcup \mathcal{J} \in \mathcal{D}_f$ by Theorem 2.9(a), so Corollary 2.10 applies. □

It is clear that $|\bigcup \mathcal{J}| \geq |\mathcal{J}|$. Example 2.12 below, which is established in detail in [4], shows that it is possible that $|\bigcup \mathcal{J}| > \lambda_f$ and $|\mathcal{J}| = \lambda_f$. Indeed, in this example, for any space Z and any non-constant function $f : Y \to Z$, the only essential set (which is then necessarily optimally esential) is the full index set I; hence $|\bigcup \mathcal{J}| = |I| > 1$, and $|\mathcal{J}| = \lambda_f = 1$. It is easily seen in this case (and it follows from Theorem 2.7) that f depends on each $i \in I$.

Example 2.12 ([4, 2.4])**.** *Let I be an index set with $0 < |I| \leq \mathfrak{c}$ and $X_I = [0, 1]^I$. There is a dense subspace Y of X_I such that for each $i \in I$ the restriction $\pi_i|Y : Y \twoheadrightarrow [0, 1]_i = [0, 1]$ is a bijection onto $[0, 1]$.*

It is reasonable to ask whether every two maximal $\mathcal{O}_f$-cellular families have the same cardinality. The following example gives a strong negative answer to that question.

Example 2.13. *Let K be an index set with $\omega \leq |K| \leq \mathfrak{c}$, $\alpha := |K|$, and $\{K(\eta) : \eta < \alpha\}$ be disjoint copies of K. Let also $I = \bigcup_{\eta<\alpha} K(\eta)$ and $X_I := \prod_{\eta<\alpha}[0,1]^{K(\eta)}$. For each $\eta < \alpha$ let Y_η be a dense subspace of $X_{K(\eta)} := [0,1]^{K(\eta)}$ such that for each $i \in K(\eta)$ the restriction $\pi_i|Y_\eta : Y_\eta \twoheadrightarrow [0,1]_i = [0,1]$ is a bijection onto $[0,1]$ (see Example 2.12). Finally, let $Y := \prod_{\eta<\alpha} Y_\eta$, $J \subseteq I$ be such that $|J \cap K_\eta| = 1$ for each $\eta < \alpha$, and $f : Y \to [0,1]^J$ be the natural projection.*

The following lemma sets forth the relevant properties of Example 2.13.

Lemma 2.14. *Let K, I, Y, J, α and f be as in Example 2.13. Then*
(a) *Y is dense in X_I, f is continuous, $|J| = \alpha$, and $J \in \mathcal{D}_f$;*
(b) *if $\eta < \alpha$, $J' \subseteq I$ is essential, and $J' \cap K(\eta) \neq \emptyset$, then $K(\eta) \subseteq J'$;*
(c) *for $\emptyset \neq S \subseteq \alpha$, the set $\bigcup_{\eta\in S} K(\eta)$ is essential; and*
(d) *for each cardinal β such that $1 \leq \beta \leq \alpha$ there is a maximal $\mathcal{O}_f$-cellular family $\mathcal{J}$ such that $|\mathcal{J}| = \beta$.*

Proof. (a) If $x, y \in Y$ and $x_J = y_J$ then $x_i = y_i$ for each $i \in J$, so $x = y$ and hence $f(x) = f(y)$. The remaining assertions are clear.

(b) If $x, y \in Y$ and $x_i \neq y_i$ (say with $i \in K(\eta)$), then each $j \in K(\eta)$ satisfies $x_j \neq y_j$. Thus for each $i \in d(x,y)$, say with $i \in K(\eta)$, we have $K(\eta) \subseteq d(x,y)$.

(c) For each $\eta < \alpha$ choose and fix distinct points $p(\eta)$, $q(\eta)$ in $X_{K(\eta)}$. (Then for each $i \in K(\eta)$ we have $p(\eta)_i \neq q(\eta)_i$.) Now given $S \subseteq \alpha$, define $x(S)$,$y(S) \in Y$ by

$x(S)_i = p(\eta)_i$ if $i \in K(\eta)$, $\eta \in S$,
$y(S)_i = q(\eta)_i$ if $i \in K(\eta)$, $\eta \in S$, and
$x(S)_i = y(S)_i = p(\eta)_i$ if $i \in K(\eta)$, $\eta \in \alpha\backslash S$.

Then $d(x(S), y(S)) = \bigcup_{\eta\in S} K(\eta)$, and for $i \in K(\eta)$ with $\eta \in S$ we have $f(x(S))_i = p(\eta)_i \neq q(\eta)_i = f(y(S))_i$.

(d) It is immediate from (b) that every essential set, since it contains one of the sets $K(\eta)$ (with $|K(\eta)| = \alpha$), is optimally essential. Now, for $1 \leq \beta \leq \alpha$ let $\{S_\zeta : \zeta < \beta\}$ be a partition of α with each $S_\zeta \neq \emptyset$, and define $J_\zeta := \bigcup\{K(\eta) : \eta \in S_\zeta\}$. Then $\mathcal{J} := \{J_\zeta : \zeta < \beta\}$ is an $\mathcal{O}_f$-cellular family with $|\mathcal{J}| = \beta$, and $\mathcal{J}$ is maximal since $\bigcup \mathcal{J} = I$. □

In relation to the construction of the set J in Example 2.13 the following more general result is valid.

Lemma 2.15. *Let $f : Y \to Z$, with $Y \subseteq X_I$, and $\mathcal{J} \in \mathfrak{J}_f$ be such that for each $J' \in \mathcal{J}$ and each index $i \in J'$ the projection $\pi_i : \pi_{J'}[Y] \to X_i$ is an injection. For each $J' \in \mathcal{J}$ fix an index $i_{J'} \in J'$ and let $J := \{i_{J'} : J' \in \mathcal{J}\}$. Then $J \in \mathcal{D}_f$ and hence $|J| = |\mathcal{J}| = \lambda_f$.*

Proof. Suppose that there exist $x, y \in Y$ such that $x_J = y_J$ and $f(x) \neq f(y)$. Since for each $J' \in \mathcal{J}$ and each index $i \in J'$ the projection $\pi_i : \pi_{J'}[Y] \to X_i$ is an injection we have $x_{J'} = y_{J'}$. Therefore $x_{\cup\mathcal{J}} = y_{\cup\mathcal{J}}$ and since $\bigcup \mathcal{J} \in \mathcal{D}_f$ by Theorem 2.9(a), we have $f(x) = f(y)$, a contradiction. Therefore $J \in \mathcal{D}_f$ and clearly $|J| = |\mathcal{J}|$.

To see that $|\mathcal{J}| = \lambda_f$ take $\mathcal{J}' \in \mathfrak{J}_f$ and let $J' \in \mathcal{J}'$. Since $\bigcup \mathcal{J} \in \mathcal{D}_f$, it follows from Theorem 2.7(c) that $J' \cap \bigcup \mathcal{J} \neq \emptyset$. Let $J'' \in \mathcal{J}$ be such that $J' \cap J'' \neq \emptyset$. If $x, y \in Y$ are such that $J' = d(x, y)$ then $J'' \subseteq d(x, y) \subseteq J'$. Thus every $J' \in \mathcal{J}'$ contains some $J'' \in \mathcal{J}$. The map $J' \to J''$ is clearly injective, so $|\mathcal{J}'| \leq |\mathcal{J}|$. □

Discussion 2.16. We showed in Theorem 2.7(c) that if $f : Y \to Z$ with $Y \subset X_I$, f depends on some J, and $J' \in \mathcal{O}_f$, then $J \cap J' \neq \emptyset$. The above example shows that the set $J \cap J'$ may fail to be essential. Further, from Theorem 2.7 it follows that if $J \subseteq I$ meets every optimally essential set then f will depend on J, and clearly $|J| \geq \lambda_f$. In the setting of Lemma 2.15 (and Example 2.13) there is $\mathcal{J}' \in \mathcal{J}_f$ such that f depends on $J \subseteq I$ iff $J \cap J' \neq \emptyset$ for each $J' \in \mathcal{J}'$. In contrast, in Example 2.1 there is a function f which depends on no J such that $|J \cap J'| = 1$ for each $J' \in \mathcal{J}$. Notice also that if the cardinality of each set $J \in \mathcal{O}_f$ is less than λ_f (which is the case in Example 2.1 if $|I| > \omega$) then each maximal $\mathcal{O}_f$-cellular family $\mathcal{J}$ satisfies $|\mathcal{J}| = \lambda_f$, for f depends on $\bigcup \mathcal{J}$ according to Theorem 2.9(a) and if $|\mathcal{J}| < \lambda_f$ then $|\bigcup \mathcal{J}| < \lambda_f$, which contradicts to Corollary 2.11. Therefore, in the setting of Example 2.1, we have $|\bigcup \mathcal{J}| = \lambda_f$.

The above observations motivate but do not answer the following question.

Question 2.17. *Does there always exist a set $J \subseteq I$ with $|J| = \lambda_f$ such that $J \in \mathcal{D}_f$?*

With the following theorem we show that the answer to Question 2.17 is in the negative.

Theorem 2.18. *Let $\kappa \geq 2^{\mathfrak{c}}$, $I := \{0,1\}^\kappa$, and $X_I := \{0,1\}^I$ with the usual product topology. Let also $\mathcal{C}$ be the set of open-and-closed subsets of I, and for $C \in \mathcal{C}$ define $x(C) \in X_I$ by $x(C)_i = 1$ if $i \in C$ and $x(C)_i = 0$ if $i \in I \backslash C$. Now, define $Y := \{x(C) : C \in \mathcal{C}\}$ and let $f := \mathrm{id} : Y \twoheadrightarrow Y = Z$. Then*

(a) *Y is dense in X_I;*

(b) *if $x(A), x(B) \in Y$, $A \neq B$, then there is $C \in \mathcal{C}$ such that $d(x(A), x(B)) = C$;*

(c) *if $\emptyset \neq J \subseteq I$, then these conditions are equivalent:*

- (i) *J is essential;*
- (ii) *J is optimally essential; and*
- (iii) *$J \in \mathcal{C}$.*

(d) *$\lambda_f = \omega$ (and, λ_f is "assumed");*

(e) *there is no $J \subseteq I$ such that $|J| = \lambda_f = \omega$ and $J \in \mathcal{D}_f$.*

Proof. It is clear that each non-empty $C \in \mathcal{C}$ satisfies $|C| = 2^\kappa$. Also, we have $\emptyset \in \mathcal{C}$, and $x(\emptyset)_i = 0$ for all $i \in I$.

(a) Let U be a basic open set in X_I. Then there exist two disjoint finite subsets F_0 and F_1 of I such that

$$U = \{x \in X_I : x|F_1 \equiv 1 \text{ and } x|F_0 \equiv 0\}.$$

Let $C \subseteq I$ be an open-and-closed set such that $F_1 \subseteq C$ and $F_0 \cap C = \emptyset$. Then $x(C) \in U \cap Y$.

(b) Let $A, B \in \mathcal{C}$ with $A \neq B$ and define $C := A \Delta B = (A \backslash B) \cup (B \backslash A)$. Then $\emptyset \neq C \in \mathcal{C}$ and $d(x(A), x(B)) = C$.

(c) (i) $\Rightarrow$ (ii). It follows from (b) that every essential set J satisfies $|J| = 2^\kappa$. Thus every essential set is optimally essential.

(ii) $\Rightarrow$ (iii). If J is (optimally) essential then there are $x(A), x(B) \in Y$ such that $d(x(A), x(B)) = J$, and from (b) we have $J = A \Delta B \in \mathcal{C}$.

(iii) $\Rightarrow$ (i). Let $\emptyset \neq J \in \mathcal{C}$. Then $d(x(\emptyset), x(J)) = J$ and $f(x(\emptyset)) = x(\emptyset) \neq x(J) = f(x(J))$, so J is essential.

(d) I is an infinite Hausdorff space, hence $\mathbf{S}(I) \geq \omega^+$ [5, 3.3(b)] and we have $\lambda_f \geq \omega$. On the other hand, being a product of separable spaces, I contains no uncountable family of pairwise disjoint, non-empty open subsets [8, 2.7.10(d)], [5, 3.9(b)]—that is, $\mathbf{S}(I) \leq \omega^+$. Hence $\lambda_f \leq \omega$.

(e) Let $J \subseteq I$ be any countable set. Since every separable Hausdorff space S satisfies $|S| \leq 2^{\mathfrak{c}}$, while $|I| = 2^\kappa \geq 2^{2^{\mathfrak{c}}} > 2^{\mathfrak{c}}$, there is $C \in \mathcal{C}$ such that $J \cap C = \emptyset$. Then $x(\emptyset)_i = x(C)_i = 0$ for all $i \in J$, so $x(\emptyset)_J = x(C)_J$, but $f(x(\emptyset)) = x(\emptyset) \neq x(C) = f(x(C))$. Thus $J \notin \mathcal{D}_f$. □

Discussion 2.19. Definition 2.3(e) suggests a natural question: Given $Y \subseteq X_I$ and $f : Y \to Z$, must there exist a (maximal) $\mathcal{O}_f$-cellular family $\mathcal{J}$ such that $|\mathcal{J}| = \lambda_f$? (In other words, can "sup" be replaced by "max" in the definition of the cardinal number λ_f?) We show in Theorem 2.24 (in ZFC) that in general the answer is "No". It is interesting that a simpler construction, similar to that of Theorem 2.18, holds in a model of ZFC with an uncountable regular limit cardinal. Here is the argument when such a cardinal is available.

Theorem 2.20. *Let $\kappa = \sup\{\kappa_\eta : \eta < \kappa\}$ be an uncountable regular limit cardinal, where $\kappa_\eta < \kappa_{\eta'}$ when $\eta < \eta' < \kappa$, let $I := \Pi_{\eta<\kappa} D(\kappa_\eta)$, and $X_I := \{0,1\}^I$. Let $\mathcal{C}$ be the set of open-and-closed subsets of I, and for $C \in \mathcal{C}$ define $x(C) \in X_I$ by $x(C)_i = 1$ if $i \in C$ and $x(C)_i = 0$ if $i \in I\backslash C$. Define $Y := \{x(C) : C \in \mathcal{C}\}$ and let $f := \mathrm{id} : Y \twoheadrightarrow Y = Z$. Then*

(a) *Y is dense in X_I;*

(b) *if $x(A), x(B) \in Y$, $A \neq B$, then there is $C \in \mathcal{C}$ such that $d(x(A), x(B)) = C$;*

(c) *if $\emptyset \neq J \subseteq I$, then these conditions are equivalent:*

(i) *J is essential;*

(ii) *J is optimally essential; and*

(iii) *$J \in \mathcal{C}$.*

(d) *$\lambda_f = \kappa$; and*

(e) *there is no $\mathcal{O}_f$-cellular family $\mathcal{J}$ such that $|\mathcal{J}| = \kappa$.*

Proof. Let $\kappa' = \Pi_{\eta<\kappa} \kappa_\eta$. (The theorem of J. König (see for example [5, 1.19]) gives $\kappa' > \kappa$, but this is not important for the present proof.) Clearly each $C \in \mathcal{C}$ satisfies $|C| = \kappa'$. The proofs of (a), (b) and (c) now follow *verbatim* the parallel proofs in Theorem 2.18, with κ' now replacing 2^κ. Clearly $\lambda_f \geq \kappa_\eta$ for each $\eta < \kappa$, so $\lambda_f \geq \kappa$. It was N. A. Shanin, using his own early version [17] of the "Δ-system" methods developed later by Erdős and Rado [9], who showed that $\mathbf{S}(I) \leq \kappa$ [19, Theorem 5]; indeed of any κ-many open sets, some κ-many have non-empty intersection [18, Theorem 5]. See [5, 3.8], [13, 5.10 and 7.6], [6, 3.28] for detailed proofs and commentary. In view of (c), Shanin's result from [18] completes the proof of (d) and gives (e). □

The existence of uncountable regular limit cardinals cannot be established in ZFC [14, 4.13], so Theorem 2.20 does not provide a definitive negative solution to the "sup = max" problem for cardinals of the form λ_f. The next theorem, which provides such a solution in ZFC, has features in common with Theorems 2.18 and 2.20, but it is necessarily more subtle: It is easily seen directly, as in [5, p. 71] or [6, 3.10], that when κ is a singular cardinal then the space $K := \Pi_{\lambda<\kappa} D(\lambda)$ can be expressed as the union of κ-many pairwise disjoint open sets (and hence $\mathbf{S}(K) = \kappa^+$).

Discussion 2.21. Here we establish notation for use in 2.22 — 2.24. In the interest of simplicity and specificity we take $\kappa_0 = \omega$, $\kappa_1 = 2^{\kappa_0}$, $\cdots$, $\kappa_{n+1} = 2^{\kappa_n}$, $\cdots$, and $\kappa = \sum_{n<\omega} \kappa_n$. (That is, $\kappa = \beth_\omega(\omega)$.) We write $K = \Pi_{n<\omega} D(\kappa_n)$. (As in the proof of Theorem 2.20 we have $|K| > \kappa$, but we do not need this fact in this proof.)

Lemma 2.22. *Let $K = \Pi_{n<\omega} D(\kappa_n)$. There is a set $I \subseteq K$ such that*
(a) *I is dense in K;*
(b) *$|I| = \kappa$;*
(c) *if $i, i' \in I$ and $i \neq i'$ then $|\{n < \omega : i_n = i'_n\}| < \omega$; and*
(d) *for each non-empty finite set $F \subset I$, there is $n = n(F) < \omega$ such that $\pi_n : K \twoheadrightarrow D(\kappa_n)$ is injective on F.*

Proof. Let D be a dense subset of K such that $|D| = \kappa$. (The existence of such D is given by the Hewitt-Marczewski-Pondiczery theorem ([5, 3.18], [13, 5.5], or [8, 2.3.15]).) Let $D = \{d(\eta) : \eta < \kappa\}$, for $\eta < \kappa$ let $n = n(\eta)$ be the least $n < \omega$ such that $\eta < \kappa_n$, and define $i(\eta) \in K$ by

$$i(\eta)_n = d(\eta)_n \text{ if } n < n(\eta),\ i(\eta)_n = \eta \text{ if } n(\eta) \leq n.$$

[We note for clarity that if $\eta = m < \omega = \kappa_0$ then $n(\eta) = 0$ and $i(\eta)_n = i(m)_n = m$ for all $n < \omega$.]

We set $I := \{i(\eta) : \eta < \kappa\}$ and we verify (a), (b), (c) and (d).

(a) It suffices to show, given $n_0 < n_1 < \cdots < n_m < \omega$ and $\eta_k < \kappa_{n_k}$ for $k \leq m$, and $V := \{x \in K : x_{n_k} = \eta_k \text{ for } k \leq m\}$, that there is $\overline{\eta} < \kappa$ such that $i(\overline{\eta}) \in V \cap I$. Since $D \cap V$ is dense in V with $|V| \geq \kappa$, we have $|D \cap V| = \kappa$. (For otherwise, say with $|D \cap V| \leq \kappa_n < \kappa$, we have the contradiction

$$\kappa \leq |V| \leq |\overline{D \cap V}^K| \leq 2^{2^{\kappa_n}} = \kappa_{n+2} < \kappa.)$$

Then there is $\overline{\eta} < \kappa$ such that $d(\overline{\eta}) \in D \cap V$ and $\overline{\eta} > \kappa_{n_m}$. Therefore $n_0 < n_1 < \cdots < n_m < n(\overline{\eta})$, so $i(\overline{\eta})_{n_k} = d(\overline{\eta})_{n_k} = \eta_k$ for $k \leq m$ and hence $i(\overline{\eta}) \in V \cap I$.

(b) From (a) follows $\kappa \leq |I| \leq |D| \leq \kappa$.

(c) Given distinct $i(\eta), i(\eta') \in I$, there is $\overline{n}$ such that $\overline{n} > n(\eta)$, $\overline{n} > n(\eta')$. Then $i(\eta)_n = \eta \neq \eta' = i(\eta')_n$ for all $n > \overline{n}$.

(d) is immediate from (c). (More directly, it is enough to choose $\overline{n} < \omega$ such that $\overline{n} > n(\eta)$ for each $i(\eta) \in F$.) □

With that preamble, we can show that cardinals of the form λ_f, defined as in Definition 2.3, may fail to be assumed.

Discussion 2.23. In Theorem 2.24 below we shall use the following notation. I is as in Lemma 2.22. For $n < \omega$ and non-empty $F \in [\kappa_n]^{<\omega}$ let $E(n, F) := \{i \in I : i_n \in F\}$. Let $\mathcal{E}_n := \{E(n, F) : \emptyset \neq F \in [\kappa_n]^{<\omega}\}$, and $\mathcal{E} := \bigcup_{n<\omega} \mathcal{E}_n$. Also, given $n < n' < \omega$, $\eta < \kappa_n$ and $F' \in [\kappa_{n'}]^{<\omega}$, set $S(n, \eta, n', F') := \{i \in I : i_n = \eta, i_{n'} \notin F'\} = E(n, \{\eta\}) \backslash E(n', F')$, and let $\mathcal{S} := \{S(n, \eta, n', F') : n < n' < \omega, \eta < \kappa_n, F' \in [\kappa_{n'}]^{<\omega}\}$.

Clearly sets in $\mathcal{E}$ and sets in $\mathcal{S}$ are open in I and hence have cardinality κ (for proof see the proof of Lemma 2.22(a)).

Theorem 2.24. *Let $X_I := \{0,1\}^I$. For $E \in \mathcal{E}$ define $x(E) \in X_I$ by $x(E)_i = 1$ if $i \in E$ and $x(E)_i = 0$ if $i \in I \backslash E$. Define $Y := \{x(E) : E \in \mathcal{E}\}$ and let $f := \mathrm{id} : Y \twoheadrightarrow Y = Z$. Then*

(a) *Y is dense in X_I;*

(b) *if $x(E_1), x(E_2) \in Y$, $E_1 \neq E_2$, then there is $S \in \mathcal{S}$ such that $S \subseteq d(x(E_1), x(E_2))$;*

(c) *every essential set is optimally essential;*

(d) *$\lambda_f = \kappa$; and*

(e) *no $\mathcal{O}_f$-cellular family satisfies $|\mathcal{J}| = \kappa$.*

Proof. (a) It is enough to show that if F_0, F_1 are disjoint finite subsets of I then there is $E \in \mathcal{E}$ such that $F_1 \subseteq E$ and $F_0 \cap E = \emptyset$ (for then $x(E) \in Y$ satisfies $x(E)_i = 1$ when $i \in F_1$, $x(E)_i = 0$ when $i \in F_0$). By Lemma 2.22(d) there is $n < \omega$ such that the restricted projection $\pi_n|I : I \to D(\kappa_n)$ is injective on $F_0 \cup F_1$, and then $E := E(n, \pi_n[F_1]) \in \mathcal{E}_n \subseteq \mathcal{E}$ is as required.

(b) We take $E_i = E(n_i, F_i)$ $(i = 1, 2)$ and we consider two cases.

<u>Case 1</u>. $n_1 = n_2$. Take arbitrary $\eta \in F_1 \Delta F_2$, say $\eta \in F_1 \backslash F_2$, let $n' > n_1$ and $\eta' \in D(\kappa_{n'})$; then $S := S(n_1, \eta, n', \{\eta'\})$ is as required.

<u>Case 2</u>. Case 1 fails, say $n_1 < n_2$. Then for arbitrary $\eta \in F_1$ the set $S := S(n_1, \eta, n_2, F_2)$ is as required.

(c) Given distinct $x(E_1), x(E_2) \in Y$ and choosing S as in (b), we have $\kappa = |I| \geq |d(x(E_1), x(E_2))| \geq |S| \geq \kappa$ since S is non-empty and open in Y, so $d(x(E_1), x(E_2)) \in \mathcal{O}_f$.

(d) For each $n < \omega$ the set $\mathcal{J}_n := \{E(n, \{\eta\}) : \eta \in \kappa_n\}$ is $\mathcal{O}_f$-cellular with $|\mathcal{J}_n| = \kappa_n$, so $\lambda_f \geq \kappa$; then since $|I| = \kappa$ we have $\lambda_f = \kappa$.

(e) In view of (a), it is enough to show $\mathbf{S}(\mathcal{S}) \leq \kappa$.

Let $\mathcal{J}$ be a cellular family for $\mathcal{S}$ and let $n < \omega$ be the minimal number for which there is a set $S(n, \eta, n', F') \in \mathcal{J}$. Let also $S(n_1, \eta_1, n_1', F_1')$ be another set in $\mathcal{J}$. Then $n_1 \geq n$. Since $S(n, \eta, n', F') \cap S(n_1, \eta_1, n_1', F_1') = \emptyset$ we have either $[n_1 = n$ and $\eta \neq \eta_1]$ or $[n_1 = n'$ and $\eta_1 \in F']$. For each $\eta_1 \in \kappa_n$ such that $\eta \neq \eta_1$ there is at most one set of the form $S(n, \eta_1, n_1', F_1')$ with $n = n_1$ in $\mathcal{J}$. Therefore, there are at most κ_n-many sets $S(n_1, \eta_1, n_1', F_1')$ in $\mathcal{J}$ such that $n_1 = n$. Also, there are $|F'| < \omega$ possible choices for $n_1 = n'$ and $\eta_1 \in F'$. And, as before, for each $\eta_1 \in F'$ there is at most one set of the form $S(n', \eta_1, n_1', F_1')$ in $\mathcal{J}$ with $n_1 = n'$. Therefore the cardinality of $\mathcal{J}$ cannot exceed $\kappa_n + |F'| = \kappa_n$. □

Discussion 2.25. (a) Constructions similar to those given in 2.21—2.24 are available when the space K is equipped with the κ-box topology. To achieve those, we need the appropriate generalization to the κ-box topology of the Hewitt-Marczewski-Pondiczery theorem ([8, 2.3.15]), together with computations of the density character and the Souslin number. For our results in this direction, see our paper [2]. To keep the present paper self-contained and coherently focused, we omit detailed statements.

(b) It is reassuring to observe that the construction given in 2.21—2.24 is not in conflict with the theorem of Erdős and Tarski [10] asserting that for every topological space X the Souslin number $\mathbf{S}(X)$ is a regular cardinal. (For a proof of that result and for some of its consequences, the reader may see [5], [13], [6].) It is immediate from this result that, in the case of the dense subspace I of $K = \Pi_{n<\omega} D(\kappa_n)$, where $\mathbf{S}(I) \geq \kappa_n$ for each $n < \omega$, the relation $\mathbf{S}(I) = \kappa$ is not possible since κ is singular. From $|I| = \kappa$ then follows $\mathbf{S}(I) = \kappa^+$. Thus in contrast with Theorem 2.24(e), which deals with the small family $\mathcal{O}_f$ of open sets, the "sup = max" question for the topology of I has a positive solution.

(c) We find it amusing, though logically inessential to the principal thrust of our paper, that in the setting of Theorem 2.24 the families $\mathcal{O}_f$ and $\mathcal{S}$, though they are not rich enough to be a base for the topology inherited by I from K, are subbases for that topology. Since we do not recall instances in the literature where families other than the usual or obvious family are shown to generate the product topology, we outline the argument in Theorem 2.26.

Since Theorem 2.26 is intended as a curiosity of stand-alone interest, we discuss the full product space $K = \Pi_{n<\omega} D(\kappa_n)$ rather than the particular dense subspace $I \subseteq K$ defined and studied in Lemma 2.22. Accordingly, for $n < \omega$ and $\emptyset \neq F \in [\kappa_n]^{<\omega}$, we write $\widetilde{E}(n, F) := \{x \in K : x_n \in F\}$, then $\widetilde{\mathcal{E}} := \{\widetilde{E}(n, F) : n < \omega, \emptyset \neq F \in [\kappa_n]^{<\omega}\}$. Similarly, for $n < n' < \omega$, $\eta < \kappa_n$ and $F' \in [\kappa_{n'}]^{<\omega}$, we write $\widetilde{S}(n, \eta, n', F') := \{x \in K : x_n = \eta, x_{n'} \notin F'\}$, then $\widetilde{\mathcal{S}} := \{\widetilde{S}(n, \eta, n', F') : n < n' < \omega, \eta < \kappa_n, F' \in [\kappa_{n'}]^{<\omega}\}$.

The function f now plays no role, so we write simply $\widetilde{\mathcal{O}} := \{\widetilde{E}_0 \triangle \widetilde{E}_1 : \widetilde{E}_i \in \widetilde{\mathcal{E}}\}$.

The product topology on K is denoted $\mathcal{T}$, and $\mathcal{B}$ is the usual (canonical) basis for $\mathcal{T}$.

Theorem 2.26. (a) *$\widetilde{\mathcal{E}}$ is a subbase for $\mathcal{T}$; and*

(b) *$\widetilde{\mathcal{O}}$ is a subbase for $\mathcal{T}$.*

Proof. It is clear that $\widetilde{\mathcal{E}} \subseteq \mathcal{T}$ and $\widetilde{\mathcal{O}} \subseteq \mathcal{T}$.

(a) Choose arbitrary $U \in \mathcal{B}$, say $U = \bigcap_{k<m} \{p \in K : p_{n_k} = \eta_k\}$ with $0 < m < \omega$ and $\eta_k < \kappa_{n_k}$ for $k < m$. It suffices to fix $p \in U$ and to show that there are $\widetilde{S}_k$ $(k < m)$ such that $p \in \bigcap_{k<m} \widetilde{S}_k \subseteq U$. For this choose $n' > \max\{n_k : k < m\}$ and arbitrary non-empty $F' \in [\kappa_{n'}]^{<\omega}$ such that $p_{n'} \notin F'$, and take $\widetilde{S}_k := \widetilde{S}(n_k, \eta_k, n', F')$.

(b) With (a) proved, it suffices to show that for each $\widetilde{S} = \widetilde{S}(n, \eta, n', F') \in \widetilde{\mathcal{S}}$ there are $F'_1, F'_2 \in [\kappa_{n'}]^{<\omega}$ such that

$$\widetilde{S} = [\widetilde{E}(n,\eta)\triangle\widetilde{E}(n',F'_1)] \cap [\widetilde{E}(n,\eta)\triangle\widetilde{E}(n',F'_2)] \tag{3}$$

(for each of the sets $\widetilde{E}(n,\eta)\triangle\widetilde{E}(n',F'_i)$ $(i = 1, 2)$ is in $\widetilde{\mathcal{O}}$). We assume without loss of generality, enlarging $F' \subseteq \kappa_{n'}$ if necessary, that $|F'| > 1$. Then (3) holds when F'_1, F'_2 are chosen non-empty so that $F'_1 \cap F'_2 = \emptyset$ and $F'_1 \cup F'_2 = F'$, for in that case, using $\widetilde{E}(n', F'_1) \cap \widetilde{E}(n', F'_2) = \emptyset$, we have

$$\begin{gathered}
[\widetilde{E}(n,\eta)\triangle\widetilde{E}(n',F'_1)] \cap [\widetilde{E}(n,\eta)\triangle\widetilde{E}(n',F'_2)] = \\
[\widetilde{E}(n,\eta)\backslash\widetilde{E}(n',F'_1)] \cap [\widetilde{E}(n,\eta)\backslash\widetilde{E}(n',F'_2)] = \\
\widetilde{S}(n,\eta,n',F'_1) \cap \widetilde{S}(n,\eta,n',F'_2) = \widetilde{S}.
\end{gathered}$$

□

We prove in Theorem 2.29 below that, under suitable conditions on X_I, Y, Z and f, every maximal $\mathcal{O}_f$-cellular family $\mathcal{J}$ satisfies not only $\bigcup \mathcal{J} \in \mathcal{D}_f$ but also $|\bigcup \mathcal{J}| = \lambda_f$. This gives a positive answer to Question 2.17 in that setting. For the proof of Theorem 2.29 we need the following lemma.

Lemma 2.27. *Let $\kappa \geq \omega$ be a cardinal number, X_I be a product space, Y be a subspace of $(X_I)_\kappa$ that contains a dense κ-invariant subspace Y', Z be a Hausdorff space, and $f \in C(Y, Z)$ be a non-constant function.*

(a) *If $J \subseteq I$ is an essential set then there exist $\widetilde{x}, \widetilde{y} \in Y' \subseteq Y$ such that $f(\widetilde{x}) \neq f(\widetilde{y})$, $|d(\widetilde{x}, \widetilde{y})| < \kappa$, and $d(\widetilde{x}, \widetilde{y}) \subseteq J$;*

(b) *if $J \subseteq I$ is such that $J \in \mathcal{O}_f$ then $|J| < \kappa$;*

(c) *if $J \subseteq I$ is an essential set with $1 < |J| < \kappa$ then for every non-empty proper subset J' of J either J' or $J \setminus J'$ contains an essential set. If, in addition, $J \in \mathcal{O}_f$ then at least one of the sets J' or $J \setminus J'$ contains an optimally essential set that has the same cardinality as J; hence J is infinite;*

(d) *if $J, J' \subseteq I$ are such that $J \in \mathcal{D}_f$ and $J' \in \mathcal{O}_f$, then $J \cap J'$ contains an optimally essential set; hence $|J \cap J'| = |J'|$; and*

(e) *if $J \subseteq I$ is such that $J \in \mathcal{D}_f$ and $\mathcal{J} := \{J_\alpha : \alpha < \beta\}$ is a maximal $\mathcal{O}_f$-cellular family then $J' := \bigcup\{J \cap J_\alpha : \alpha < \beta\} \in \mathcal{D}_f$.*

Proof. (a) Let $x, y \in Y$ be such that $f(x) \neq f(y)$ and $d(x, y) = J$. Since Z is a Hausdorff space and f is continuous, there are disjoint basic open neighborhoods U and V in $(X_I)_\kappa$ of x and y, respectively, such that $f[U \cap Y] \cap f[V \cap Y] = \emptyset$. Without loss of generality, we can assume (shrinking U_i and V_i, if necessary) that $U_i = V_i$ whenever $x_i = y_i$. Since Y' is dense in Y there exist $x' \in U \cap Y'$ and $y' \in V \cap Y'$. Clearly, $f(x') \neq f(y')$. Now define $\widetilde{x} := x'$ and define $\widetilde{y} \in X_I$ as follows:

$$\widetilde{y}_i = y'_i \text{ if } i \in R(U) \cup R(V) \text{ and } x_i \neq y_i, \text{ and}$$
$$\widetilde{y}_i = x'_i \text{ if } x_i = y_i \text{ or } i \in I \backslash (R(U) \cup R(V)).$$

Then from $|R(U) \cup R(V)| < \kappa$ we have $\widetilde{x}, \widetilde{y} \in Y' \subseteq Y$, $|d(\widetilde{x}, \widetilde{y})| < \kappa$, and $d(\widetilde{x}, \widetilde{y}) \subseteq J$. Since $\widetilde{x} = x' \in U$ and $\widetilde{y}_i \in V_i$ whenever $i \in R(V)$ we have $\widetilde{y} \in V$. Therefore $f(\widetilde{x}) \neq f(\widetilde{y})$.

(b) Suppose there is an optimally essential $J \subseteq I$ such that $|J| \geq \kappa$ and let $x, y \in Y$ be such that $f(x) \neq f(y)$ and $d(x, y) = J$. Let $\widetilde{x}, \widetilde{y} \in Y'$ be as in (a). Then $f(\widetilde{x}) \neq f(\widetilde{y})$, $d(\widetilde{x}, \widetilde{y}) \subseteq J$, and $|d(\widetilde{x}, \widetilde{y})| < \kappa \leq |J|$, contrary to the fact that $J \in \mathcal{O}_f$.

(c) It follows from (a) that there exist $\widetilde{x}, \widetilde{y} \in Y'$ such that $f(\widetilde{x}) \neq f(\widetilde{y})$ and $d(\widetilde{x}, \widetilde{y}) \subseteq J$. Let J' be a non-empty proper subset of J. We define $w \in X_I$ by $w_i = \widetilde{x}_i$ if $i \in I \setminus J'$ and $w_i = \widetilde{y}_i$ if $i \in J'$. It is clear that $w \in Y' \subseteq Y$ and therefore $f(w)$ is well-defined. If $f(w) \neq f(\widetilde{x})$ then J' contains an essential set since $d(\widetilde{x}, w) \subseteq J'$, and if $f(w) = f(\widetilde{x})$ then $f(w) \neq f(\widetilde{y})$ and then $J \setminus J'$ contains an essential set since $\emptyset \neq d(\widetilde{y}, w) \subseteq J \setminus J'$. Clearly an optimally essential set with a proper essential subset cannot be finite.

(d) It follows from Theorem 2.7 that $J \cap J' \neq \emptyset$. If $J' \setminus J = \emptyset$ then there is nothing to prove. If $J' \setminus J \neq \emptyset$ then, according to (c), either $J' \setminus J$ or $J \cap J'$ contains an essential set, and since J intersects every essential set, we conclude that $J \cap J'$ contains an essential set and therefore it contains an optimally essential set. Hence, $|J \cap J'| = |J'|$.

(e) Let J and $\mathcal{J} := \{J_\alpha : \alpha < \beta\}$ be as hypothesized. Let also $J' := \bigcup \{J \cap J_\alpha : \alpha < \beta\}$. It follows from (d) that each non-empty set $J \cap J_\alpha$ contains an optimally essential set; hence J' contains a cellular family of optimally essential sets.

Suppose that $J' \notin \mathcal{D}_f$. Then there exist $x, y \in Y$ such that $x_{J'} = y_{J'}$ and $f(x) \neq f(y)$. Let $K \subseteq d(x, y)$ be an optimally essential set (Proposition 2.5). Then, according to Theorem 2.7(c), $K' := K \cap J \neq \emptyset$ and from (d) we conclude that K' contains an optimally essential set K''. Clearly $K'' \cap J_\alpha = \emptyset$ for each $\alpha < \beta$, contrary to the maximality of $\mathcal{J}$. Therefore f depends on J'. $\square$

Remark 2.28. In the case when $Y' = \Sigma_\kappa(p)$ for some $p \in X_I$ then in Lemma 2.27(d) the set $J \cap J'$ is itself optimally essential; in that case, Lemma 2.27(e) may be restated as follows: Every set $J \subseteq I$ with $J \in \mathcal{D}_f$ contains a subset J' such that $J' \in \mathcal{D}_f$ and J' is a union of a cellular family of optimally essential sets (compare with Theorem 2.9(a)).

As a corollary of the above lemma we obtain the following result.

Theorem 2.29. *Let $\kappa \geq \omega$ be a cardinal number, X_I be a product space, Y be a subspace of $(X_I)_\kappa$ that contains a dense (in Y) κ-invariant subspace Y', Z be a Hausdorff space, and $f \in C(Y,Z)$ be a non-constant function. If $\mathcal{J} \in \mathfrak{J}_f$ then*
(a) *if $\kappa \leq \lambda_f$ then $|\bigcup \mathcal{J}| = \lambda_f$;*
(b) *if $\lambda_f \leq \kappa$ then $|\bigcup \mathcal{J}| \leq \kappa$; and*
(c) *if $\lambda_f < \mathrm{cf}(\kappa)$ then $|\bigcup \mathcal{J}| < \kappa$.*

Proof. According to Lemma 2.27, if $J \in \mathcal{J}$ then $|J| < \kappa$.
(a) If $\kappa \leq \lambda_f$ then $|\bigcup \mathcal{J}| = |\bigcup\{J_\alpha : \alpha < |\mathcal{J}|\}| \leq \kappa|\mathcal{J}| \leq \kappa\lambda_f = \lambda_f$. It follows from Corollary 2.11 that $|\bigcup \mathcal{J}| \geq \lambda_f$; hence $|\bigcup \mathcal{J}| = \lambda_f$.
(b) We have $|\mathcal{J}| \leq \lambda_f \leq \kappa$ and $|J| < \kappa$ for each $J \in \mathcal{J}$, so $|\bigcup \mathcal{J}| \leq \kappa$.
(c) We have $|\mathcal{J}| \leq \lambda_f < \mathrm{cf}(\kappa)$ and $|J| \leq \kappa$ for each $J \in \mathcal{J}$, so $|\bigcup \mathcal{J}| < \kappa$. □

If in Theorem 2.29(a) more information is available about λ_f and κ then stronger conclusions are available.

Theorem 2.30. *Let $\kappa \geq \omega$ be a cardinal number, X_I be a product space, Y be a subspace of $(X_I)_\kappa$ that contains a dense (in Y) κ-invariant subspace Y', Z be a Hausdorff space, and $f \in C(Y,Z)$. If $\lambda_f > \kappa$, or $\lambda_f = \kappa$ and κ is regular, then*
(a) *there is an $\mathcal{O}_f$-cellular family $\mathcal{J}$ such that $|\mathcal{J}| = \lambda_f$;*
(b) *$|\mathcal{J}| = |\mathcal{J}'|$ whenever $\mathcal{J}, \mathcal{J}' \in \mathfrak{J}_f$.*

Proof. Let $\lambda_f > \kappa$, or $\lambda_f = \kappa$ with κ regular.
(a) Suppose that $|\mathcal{J}| < \lambda_f$ for each $\mathcal{O}_f$-cellular family $\mathcal{J}$. Then there exists a set $\{\mathcal{J}_\delta : \delta < \gamma\}$ of maximal $\mathcal{O}_f$-cellular families such that $\lambda_f = \sup\{|\mathcal{J}_\delta| : \delta < \gamma\}$ and $|\mathcal{J}_{\delta_1}| < |\mathcal{J}_{\delta_2}|$ whenever $\delta_1 < \delta_2 < \gamma$. Let $\mathcal{J}_{\delta_1}$ be one of these families and let $J := \bigcup \mathcal{J}_{\delta_1}$. In the case when $\lambda_f > \kappa$ we choose $\mathcal{J}_{\delta_1}$ to be such that $|\mathcal{J}_{\delta_1}| > \kappa$. Then $J \in \mathcal{D}_f$ by Theorem 2.9(a). Since the cardinality of each optimally essential set is $< \kappa$ (Lemma 2.27) we have $|J| \leq \kappa|\mathcal{J}_{\delta_1}| = |\mathcal{J}_{\delta_1}|$ in the case $\lambda_f > \kappa$, and we have $|J| < \kappa$ in the case $\lambda_f = \kappa$ and κ is regular. In either case there exists $\delta_2 > \delta_1$ such that $|J| < |\mathcal{J}_{\delta_2}|$. According to Theorem

2.7(c), $K \cap J \neq \emptyset$ for each $K \in \mathcal{J}_{\delta_2}$ and since the elements of $\mathcal{J}_{\delta_2}$ are pairwise disjoint sets, we get a contradiction with $|J| < |\mathcal{J}_{\delta_2}|$.

(b) According to (a) there is a (maximal) $\mathcal{O}_f$-cellular family $\mathcal{J}$ such that $|\mathcal{J}| = \lambda_f$. Suppose that there is another maximal $\mathcal{O}_f$-cellular family $\mathcal{J}'$ such that $|\mathcal{J}'| < \lambda_f$. Then $J := \bigcup \mathcal{J}' \in \mathcal{D}_f$ by Theorem 2.9(a) and $|J| < \lambda_f$. According to Theorem 2.7(c), $K \cap J \neq \emptyset$ for each $K \in \mathcal{J}$ and since the elements of $\mathcal{J}$ are pairwise disjoint sets, we get a contradiction with $|J| < \lambda_f = |\mathcal{J}|$. □

Remark 2.31. To see that when $\lambda_f < \kappa$ we cannot always make the same conclusion as in Theorem 2.30(b), consider Example 2.1 with $|I| = \omega$ and $\kappa = \omega_1$. There, $\lambda_f = \omega$ and there are maximal $\mathcal{O}_f$-cellular families of every positive cardinality $\leq \omega$. For each such family $\mathcal{J}$ we have $|\bigcup \mathcal{J}| = \omega$, which agrees with Theorem 2.29(c).

For some applications of Theorems 2.29 and 2.30 see [1].

In the special case $\kappa = \omega$ of Lemma 2.27, the optimally essential sets can be identified in concrete form.

Lemma 2.32. *Let X_I be a product space, Y be a subspace of X_I that contains a dense (in Y) ω-invariant subspace Y', Z be a Hausdorff space, and $f \in C(Y, Z)$ be a non-constant function. If $J \subseteq I$ is optimally essential, then $|J| = 1$. Therefore $J_f = \bigcup \mathcal{J}$ for every $\mathcal{J} \in \mathfrak{J}_f$. Thus, $\mathcal{J} := \{\{i\} : i \in J_f\}$ is the only maximal $\mathcal{O}_f$-cellular family, $J_f \in \mathcal{D}_f$, and if $J' \subset I$ is such that $J' \in \mathcal{D}_f$ then $J_f \subseteq J'$.*

Proof. Let $J \subset I$ be an optimally essential set. It follows from Lemma 2.27(b) that J is finite. Let $x, y \in Y$ be such that $f(x) \neq f(y)$ and $J = d(x, y)$. According to Lemma 2.27(a) there exist points $\widetilde{x}, \widetilde{y} \in Y' \subseteq Y$ such that $f(\widetilde{x}) \neq f(\widetilde{y})$ and $d(\widetilde{x}, \widetilde{y}) \subseteq J$. Since $J \in \mathcal{O}_f$ and is finite we have $d(\widetilde{x}, \widetilde{y}) = J$. Assume that $|J| > 1$. Then it follows from Lemma 2.27(c) that $|J| \geq \omega$, which is a contradiction. Therefore $|J| = 1$, hence $J_f = \bigcup \mathcal{J}$. The final assertions, that $J_f \in \mathcal{D}_f$ and that $J_f \subseteq J'$ whenever $J' \in \mathcal{D}_f$ follow directly from Theorem 2.9 and Theorem 2.7. □

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