HAMILTONIAN LACEABILITY OF HYPERCUBES WITH FAULTS OF CHARGE ONE

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ABSTRACT. In 2007, in their paper *Path coverings with prescribed ends in faulty hypercubes*, N. Castañeda and I. Gotchev formulated the following conjecture: Let n and k be positive integers with $n \geq k + 3$ and $\mathcal{F}$ be a set of k even (odd) and $k + 1$ odd (even) vertices in the binary hypercube Q_n. If u_1 and u_2 are two distinct even (odd) vertices in $\mathcal{Q}_n - \mathcal{F}$ then for $\mathcal{Q}_n - \mathcal{F}$ there exists a Hamiltonian path that connects u_1 to u_2. In the same paper Castañeda and Gotchev proved that conjecture for $k = 1$ and $k = 2$. Here we provide a proof for $k = 3$.

1. INTRODUCTION

The n-dimensional hypercube $\mathcal{Q}_n$ is the graph with vertex set $\mathcal{V}(\mathcal{Q}_n)$ that consists of all binary sequences of length n and with edge set $\mathcal{E}(\mathcal{Q}_n)$ that consists of all pairs of binary sequences that differ in exactly one position. Due to the applications of hypercubes as prospective interconnection networks for parallel and distributed computing, in the recent years much attention has been given to the problem of finding Hamiltonian cycles or maximal cycles in $\mathcal{Q}_n$ with or without some faulty vertices and/or faulty edges (see [HHW], [L], [T], [LW], [LS], [F1], [F2], [D], [CK2], [CG1]).

One way to construct a Hamiltonian cycle (path) for $\mathcal{Q}_n$, with a set $\mathcal{F}$ of faulty vertices, is the following: split $\mathcal{Q}_n$ in a natural way into two $n - 1$ dimensional hypercubes $\mathcal{Q}_n^{top}$ and $\mathcal{Q}_n^{bot}$; find appropriate path coverings of the vertices of $\mathcal{Q}_n^{top} - \mathcal{F}^{top}$ and $\mathcal{Q}_n^{bot} - \mathcal{F}^{bot}$ by simple paths with prescribed end vertices such that every vertex of the induced graph is contained in exactly one of the paths; connect the corresponding end vertices of the paths with bridges between $\mathcal{Q}_n^{top}$ and $\mathcal{Q}_n^{bot}$ to form the required Hamiltonian cycle (path).

Therefore it is useful to know when for $\mathcal{Q}_n$, with a set $\mathcal{F}$ of faulty vertices, there exists a path covering of $\mathcal{Q}_n - \mathcal{F}$ by simple paths with prescribed ends such that every vertex of $\mathcal{Q}_n - \mathcal{F}$ is contained in exactly one of the paths. This and

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similar questions for different graphs have been studied by many authors (see [EO], [PKL], [CK1], [DG]).

N. Castañeda and I. Gotchev in [CG2] posed the following question: What is the minimal dimension m of the hypercube $\mathcal{Q}_m$ such that for every $n \geq m$ and every set $\mathcal{F}$ of $M \geq 0$ faulty vertices from $\mathcal{Q}_n$ such that the absolute value of the difference of the numbers of the faulty vertices of the two parities (called *charge* of the fault $\mathcal{F}$) is C, there exists a path covering of $\mathcal{Q}_n - \mathcal{F}$ with N paths whose end vertices are with different parity and O paths whose end vertices are of the same parity, where all of the end vertices of these paths belong to an arbitrary set of non-faulty vertices. Brief explanation of the exact meaning of these words can be found in Section 2 where more precise definitions are given including the definition of the symbol $[M, C, N, O]$ that represents the number m mentioned above.

In the same article [CG2] N. Castañeda and I. Gotchev formulated the following conjecture:

Conjecture 1.1. *Let n and k be positive integers with $n \geq k+3$ and $\mathcal{F}$ be a set of k even (odd) and $k + 1$ odd (even) vertices in the binary hypercube Q_n. If u_1 and u_2 are two distinct even (odd) vertices in $\mathcal{Q}_n - \mathcal{F}$ then for $\mathcal{Q}_n - \mathcal{F}$ there exists a Hamiltonian path that connects u_1 to u_2.*

Using the notation mentioned above Conjecture 1.1 is equivalent to the statement that $[2k + 1, 1, 0, 1] = k + 3$ for every $k \geq 1$. In [CG2] the authors proved that conjecture for $k = 1$ and $k = 2$. In terms of laceability, Conjecture 1.1 says that $\mathcal{Q}_n - \mathcal{F}$ is Hamiltonian laceable for every fault of charge one and cardinality $2k + 1$ if the dimension of the cube is $n \geq k + 3$.

In this paper we prove the above conjecture for $k = 3$, or equivalently, we show that $[7, 1, 0, 1] = 6$. For that proof, together with many other results from [CG2], we need the fact that $[4, 0, 2, 0] = 5$ which we prove as a preliminary lemma.

2. SOME DEFINITIONS AND NOTATIONS

A *path covering* of a graph $G = (V, E)$ is a set of simple paths in G with the property that every vertex of G is contained in exactly one of the paths. A $k-$path covering of G is a path covering consisting of k paths. We are interested in $k-$path coverings of induced subgraphs of the $n-$dimensional binary hypercube $\mathcal{Q}_n$, where $k \geq 1$. An induced subgraph of $\mathcal{Q}_n$ is also known as a *faulty hypercube*.

A vertex in $\mathcal{Q}_n$ is *even* if it has an even number of 1s and *odd* otherwise. For convenience, we call the vertices of one parity *red* and the vertices of the opposite parity *green*. A *fault* $\mathcal{F}$ in $\mathcal{Q}_n$ is a set of faulty/deleted vertices.

If a, b are two vertices in $\mathcal{Q}_n$ then by $d_H(a, b)$ we denote the *Hamming distance* between a and b, i.e. the number of components where a and b differ.

The following terminology was introduced in [CG2]. The *mass M* of a fault $\mathcal{F}$ is the number of vertices in the fault. The *charge C* of a fault is the absolute value of the difference between the number of red vertices and the number of green vertices. A fault is called *neutral* if its charge is zero. When the endpoints of a path are of the same parity the path is called *charged*. Otherwise, the path is called *neutral*. Regarding a pair of vertices: a given pair of vertices is called *charged* if the two vertices are of the same parity and it is called *neutral* if they are of opposite parity. If the two elements of a charged pair of vertices are red (green) the pair is called *red* (*green*).

Let $r(\mathcal{F})$ be the number of red vertices and $g(\mathcal{F})$ be the number of green vertices in a fault $\mathcal{F}$ of $\mathcal{Q}_n$. Let also $\mathcal{E}$ be a set of disjoint pairs of vertices of $\mathcal{Q}_n$, $r(\mathcal{E})$ be the number of red pairs in $\mathcal{E}$, and $g(\mathcal{E})$ be the number of green pairs in $\mathcal{E}$. *The set of pairs $\mathcal{E}$ is in balance with the fault $\mathcal{F}$ if all the vertices in the elements of $\mathcal{E}$ are from $\mathcal{Q}_n - \mathcal{F}$ and $r(\mathcal{F}) - g(\mathcal{F}) = g(\mathcal{E}) - r(\mathcal{E})$.* Since $\mathcal{Q}_n$ is a bipartite graph, a necessary condition for a set $\mathcal{E}$ of pairs of vertices to be the set of endpoints of a path covering of $\mathcal{Q}_n - \mathcal{F}$ is that $\mathcal{E}$ to be in balance with $\mathcal{F}$.

The following definition can be found in [CG2].

Definition 2.1. *Let M, C, N, O be nonnegative integers and $\mathcal{F}$ be a fault of mass M and charge C in $\mathcal{Q}_n$. We say that one can freely prescribe ends for a path covering of $\mathcal{Q}_n - \mathcal{F}$ with N neutral paths and O charged paths if*

   (i) *there exists at least one set $\mathcal{E}$ of disjoint pairs of vertices that is in balance with $\mathcal{F}$ and contains exactly N neutral pairs and O charged pairs; and*
   (ii) *for every set $\mathcal{E}$ of disjoint pairs of vertices that is in balance with $\mathcal{F}$ and contains exactly N neutral pairs and O charged pairs there exists a path covering of $\mathcal{Q}_n - \mathcal{F}$ such that the set of pairs of end vertices of the paths in the covering coincides with $\mathcal{E}$.*

It is easy to see that if in $\mathcal{Q}_n$ there exists a fault $\mathcal{F}$ of mass M and charge C, and a set of pairs of vertices $\mathcal{E}$ that is in balance with $\mathcal{F}$ and contains exactly N neutral pairs and O charged pairs, then

$$2^n \geq M + C + 2N + 2O.$$

Let $\mathcal{A}_{M,C,N,O}$ be the set of nonnegative integers m such that

   (i) $m \geq \log_2 [M + C + 2N + 2O]$; and
   (ii) for every $n \geq m$ and for every fault $\mathcal{F}$ of mass M and charge C in $\mathcal{Q}_n$ one can freely prescribe ends for a path covering of $\mathcal{Q}_n - \mathcal{F}$ with N neutral paths and O charged paths.

We let $[M, C, N, O]$ denote the smallest element in $\mathcal{A}_{M,C,N,O}$ if this set is non-empty. For example, the statement $[0, 0, 1, 0] = 1$ is the so called Havel's lemma ([H],[D]), which states that given any two vertices of opposite parity in $\mathcal{Q}_n$, with $n \geq 1$, there exists a Hamiltonian path with these two vertices as endpoints. The statement $[1, 1, 0, 1] = 2$ was proven by Lewinter and Widulski [LW] and says that $\mathcal{Q}_n$ with any vertex removed is Hamiltonian laceable. The next step in establishing exact values for $[M, C, N, O]$ was made by Dvořák who proved the following lemma:

Lemma 2.2 ([D]). *Let $n \geq 2$, a_1, a_2 be two distinct vertices of the same parity, and b_1, b_2 be two distinct vertices of the opposite parity in the hypercube $\mathcal{Q}_n$. Then there exist two disjoint simple paths, one joining a_1 to b_1 and the other joining a_2 to b_2, such that each vertex of $\mathcal{Q}_n$ is contained in one of these paths.*

Dvořák's lemma is precisely the statement $[0, 0, 2, 0] = 2$.

The following table summarizes some of the results about path coverings of the (faulty) hypercube that are known to us. Most of them are contained in [CG2]. The rows represent admissible combinations of M and C and the columns contain all the values of N and O such that $N + O \leq 3$. Each star in the table represents an impossible case. The missing entries in the table correspond to values of $[M, C, N, O]$ that we do not know yet. The inequalities in the table represent an upper or lower bound of the corresponding entry. The values with an asterisk represent results known to us that have not been published yet.

$MC\backslash NO$	01	10	20	11	02	30	21	12	03
00	$\star$	1	2	$\star$	4	5	$\star$	4^*	$\star$
11	2	$\star$	$\star$	4	$\star$	$\star$	5^*	$\star$	5^*
20	$\star$	4	4	$\star$	5		$\star$		$\star$
22	$\star$	$\star$	$\star$	$\star$	4	$\star$	$\star$	≤ 6	$\star$
31	4	$\star$	$\star$	5	$\star$	$\star$		$\star$	
33	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	≤ 6
40	$\star$	5	5^*	$\star$			$\star$		$\star$
42	$\star$	$\star$	$\star$	$\star$	5	$\star$	$\star$		$\star$
44	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
51	5	$\star$	$\star$	≥ 5	$\star$	$\star$		$\star$	

In the present paper we provide a proof that $[4, 0, 2, 0] = 5$ and we use it to prove that $[7, 1, 0, 1] = 6$. The fact that $[0, 0, 1, 2] = 4$ is proved in [CGGL]. The proofs of the statements $[1, 1, 2, 1] = 5$ and $[1, 1, 0, 3] = 5$ will appear in a forthcoming paper.

In the proofs that follow we shall use the notation already introduced in [CG2]. The hypercube $\mathcal{Q}_{n+1}$ is viewed as two copies of the $n-$dimensional hypercube

which are called *top plate* and *bottom plate* and are denoted by $\mathcal{Q}_{n+1}^{top}$ and $\mathcal{Q}_{n+1}^{bot}$, respectively. The edges connecting the two plates are called *bridges*. The hypercube $\mathcal{Q}_n$ is identified with the group $\mathbf{Z}_2^n$. We view $\mathcal{Q}_n$ as a Cayley graph with the standard system of generators

$$\mathbf{S} = \{e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)\}.$$

An oriented edge in $\mathcal{Q}_n$ is represented by (a, x), where a is the starting vertex and x is an element from the system of generators $\mathbf{S}$. A path is represented by (a, ω), where a is the initial vertex and ω is a word with letters from $\mathbf{S}$. If $\omega = x_1, x_2, \ldots, x_k$ then the path (a, ω) is the path $a, ax_1, ax_1x_2, \ldots, ax_1x_2 \cdots x_n$. When the second endvertex of the path (a, ω) is known, say b, then we write $(a, \omega; b)$. The algebraic content of a word ω is the element of $\mathbf{Z}_2^n$ that is obtained by multiplying all the letters of ω. A path (a, ω) is simple if no subword of ω is algebraically equivalent to the identity $(0, 0, \ldots, 0)$. A path (a, ω) is a cycle if ω is algebraically equivalent to the identity but no proper subword of ω is algebraically equivalent to the identity.

We shall also use the following notation: ω^R means the reverse word of ω; ω' denotes the word obtained after the last letter is deleted from ω; ω^* is the word obtained after the first letter is deleted from ω; $\varphi(\omega)$ is the first letter of ω, and $\lambda(\omega)$ is the last letter of ω. The length of the word ω is denoted by $|\omega|$. The letter v shall be reserved for steps connecting two plates. The letters $x, y, \ldots$ shall be reserved to represent steps along the plates.

3. $[4, 0, 2, 0] = 5$ AND $[7, 1, 0, 1] = 6$

Lemma 3.1 ($[4, 0, 2, 0] = 5$)**.** *Let $n \geq 5$, r_1, r_2, r_3, r_4 be four distinct red vertices and g_1, g_2, g_3, g_4 be four distinct green vertices in $\mathcal{Q}_n$. Then there exists a $2-$path covering $(r_3, \xi; g_3)$, $(r_4, \eta; g_4)$ of $\mathcal{Q}_n - \mathcal{F}$, where $\mathcal{F} = \{r_1, g_1, r_2, g_2\}$. The claim is not true for $n = 4$.*

Proof. Let $n = 4$, $r = (0, 1, 0, 0)$, $r_1 = (1, 1, 0, 1)$, $r_2 = (1, 0, 0, 0)$, $r_3 = (1, 1, 1, 0)$, $g = (1, 0, 0, 1)$, $g_1 = (1, 1, 1, 1)$, $g_2 = (1, 0, 1, 0)$ and $g_3 = (0, 0, 1, 1)$ be vertices in $\mathcal{Q}_4$. Then one can directly verify that a path covering $(r, \xi; g)$, $(r_1, \eta; g_1)$ of $\mathcal{Q}_4 \setminus \{r_2, r_3, g_2, g_3\}$ does not exist.

Let $n \geq 5$. We split $\mathcal{Q}_n$ in some direction that separates r_1 from r_2. Without loss of generality we can assume that $r_1 \in \mathcal{Q}_n^{top}$ and $r_2 \in \mathcal{Q}_n^{bot}$. There are two cases to consider: **Case A**: Each plate contains one deleted green vertex; **Case B**: The two deleted green vertices are on the same plate that we may assume to be $\mathcal{Q}_n^{top}$.

Case A. $g_1 \in \mathcal{Q}_n^{top}, g_2 \in \mathcal{Q}_n^{bot}$.

A1. $r_3, r_4, g_3, g_4 \in \mathcal{Q}_n^{top}$.

Since $[2, 0, 2, 0] = 4$, there exists a path covering $(r_3, \xi; g_3)$, $(r_4, \eta; g_4)$ of $\mathcal{Q}_n^{top} - \{r_1, g_1\}$. Without loss of generality we can assume that $|\xi| \geq |\eta|$. Therefore, there are words μ and ν such that $\xi = \mu\nu$ and $\{r_3\mu v, r_3\mu' v\} \cap \{r_2, g_2\} = \emptyset$. Since $[2, 0, 1, 0] = 4$, there exists a Hamiltonian path $(r_3\mu' v, \zeta; r_3\mu v)$ of $\mathcal{Q}_n^{bot} - \{r_2, g_2\}$. The desired 2−path covering of $\mathcal{Q}_n - \mathcal{F}$ is $(r_3, \mu' v\zeta v\nu; g_3)$, $(r_4, \eta; g_4)$.

A2. $r_3, r_4, g_3 \in \mathcal{Q}_n^{top}$, $g_4 \in \mathcal{Q}_n^{bot}$.

Let $g^* \neq g_1, g_3$ be a green vertex in $\mathcal{Q}_n^{top}$ such that $g^* v \neq r_2$. Since $[2, 0, 2, 0] = 4$, there exists a 2−path covering $(r_3, \xi; g_3)$, $(r_4, \eta; g^*)$ of $\mathcal{Q}_n^{top} - \{r_1, g_1\}$. Also, since $[2, 0, 1, 0] = 4$, there exists a Hamiltonian path $(g^* v, \zeta; g_4)$ of $\mathcal{Q}_n^{bot} - \{r_2, g_2\}$. The desired 2−path covering of $\mathcal{Q}_n - \mathcal{F}$ is $(r_3, \xi; g_3)$, $(r_4, \eta v\zeta; g_4)$.

A3. $r_3, g_3 \in \mathcal{Q}_n^{top}$; $r_4, g_4 \in \mathcal{Q}_n^{bot}$.

Since $[2, 0, 1, 0] = 4$, there exists a Hamiltonian path $(r_3, \xi; g_3)$ of $\mathcal{Q}_n^{top} - \{r_1, g_1\}$ and a Hamiltonian path $(r_4, \eta; g_4)$ of $\mathcal{Q}_n^{bot} - \{r_2, g_2\}$. The desired 2−path covering of $\mathcal{Q}_n - \mathcal{F}$ is $(r_3, \xi; g_3)$, $(r_4, \eta; g_4)$.

A4. $r_3, g_4 \in \mathcal{Q}_n^{top}$; $r_4, g_3 \in \mathcal{Q}_n^{bot}$.

Since $[2, 0, 1, 0] = 4$, there exists a Hamiltonian path $(r_3, \xi; g_4)$ of $\mathcal{Q}_n^{top} - \{r_1, g_1\}$. We can find words μ and ν, with $\xi = \mu\nu$, $\{r_3\mu v, r_3\mu' v\} \cap \{r_2, g_2, r_4, g_3\} = \emptyset$ and $|\mu|$ odd for $|\xi| \geq 13$. Also, since $[2, 0, 2, 0] = 4$, there exists a 2−path covering $(r_3\mu' v, \eta; g_3)$, $(r_4, \zeta; r_3\mu v)$ of $\mathcal{Q}_n^{bot} - \{r_2, g_2\}$. The desired 2−path covering of $\mathcal{Q}_n - \mathcal{F}$ is $(r_3, \mu' v\eta; g_3)$, $(r_4, \zeta v\nu; g_4)$.

A5. $r_3, r_4 \in \mathcal{Q}_n^{top}$, , $g_3, g_4 \in \mathcal{Q}_n^{bot}$.

Let g_5, g_6 be two distinct green vertices in $\mathcal{Q}_n^{top} - \{g_1\}$ such that $g_5 v \neq r_2$ and $g_6 v \neq r_2$. Since $[2, 0, 2, 0] = 4$, there exists a 2−path covering $(r_3, \xi; g_5)$, $(r_4, \eta; g_6)$ of $\mathcal{Q}_n^{top} - \{r_1, g_1\}$ and a 2− path covering $(g_5 v, \zeta; g_3)$, $(g_6 v, \theta; g_4)$ of $\mathcal{Q}_n^{bot} - \{r_2, g_2\}$. The desired 2−path covering of $\mathcal{Q}_n - \mathcal{F}$ is $(r_3, \xi v\zeta; g_3)$, $(r_4, \eta v\theta; g_4)$.

Case B. $r_1, g_1, g_2 \in \mathcal{Q}_n^{top}$; $r_2 \in \mathcal{Q}_n^{bot}$.

The terminals r_3, r_4, g_3, g_4 can be distributed in 16 different ways among the plates. However, some cases are equivalent to others thanks to the symmetry. There is a total of 10 nonequivalent subcases.

B1. $r_3, r_4, g_3, g_4 \in \mathcal{Q}_n^{top}$.

There exists a Hamiltonian path $(r_3, \xi; r_4)$ of $\mathcal{Q}_n^{top} - \{r_1, g_1, g_2\}$ for $[3, 1, 0, 1] = 4$. There are two possibilities (1) $\xi = \mu\nu\eta$ with $r_3\mu = g_3$, $g_3\nu = g_4$ and (2) $\xi = \mu\nu\eta$ with $r_3\mu = g_4$, $g_4\nu = g_3$. For case (1), since $[1, 1, 0, 1] = 2$, there exists a Hamiltonian path $(r_3\mu' v, \theta; g_3\nu' v)$ of $\mathcal{Q}_n^{bot} - \{r_2\}$. The desired 2−path covering of $\mathcal{Q}_n - \mathcal{F}$ is $(r_3, \mu' v\theta v\nu\nu'^R; g_3)$, $(r_4, \eta^R; g_4)$. For case (2) we can assume, without loss of generality, that $g_4 v \neq r_2$. Let $(r_3\mu' v, \theta; g_3(\nu^R)' v)$,

$(r_4(\eta^R)'v, \gamma; g_4 v)$ be a 2−path covering of $Q_n^{bot} - \mathcal{F}$. Such path covering exists since $[1,1,1,1] = 4$. Then the desired 2−path covering of $\mathcal{Q}_n - \mathcal{F}$ is $(r_3, \mu'v\theta v\nu^*; g_3), (r_4, (\eta^R)'v\gamma v; g_4)$.

B2. $r_3, r_4, g_3 \in \mathcal{Q}_n^{top}; g_4 \in \mathcal{Q}_n^{bot}.$

There exists a Hamiltonian path $(r_3, \xi; r_4)$ of $\mathcal{Q}_n^{top} - \{r_1, g_1, g_2\}$, where $\xi = \mu\nu$, with $r_3\mu = g_3$, for $[3,1,0,1] = 4$. If $r_4(\nu^R)'v \neq g_4$ then there exists a Hamiltonian path $(r_4(\nu^R)'v, \theta; g_4)$ of $\mathcal{Q}_n^{bot} - \{r_2\}$ for $[1,1,0,1] = 2$. The desired 2−path covering of $\mathcal{Q}_n - \mathcal{F}$ is $(r_3, \mu; g_3), (r_4, (\nu^R)'v\theta; g_4)$.

Now, let $r_4(\nu^R)'v = g_4$. If $g_3v \neq r_2$ then there exists a Hamiltonian path $(r_3\mu'v, \theta; g_3v)$ of $\mathcal{Q}_n^{bot} - \{r_2, g_4\}$ for $[2,0,1,0] = 4$. The desired 2−path covering of $\mathcal{Q}_n - \mathcal{F}$ is $(r_3, \mu'v\theta v; g_3), (r_4, (\nu^R)'v; g_4)$.

Finally, let $g_3v = r_2$ and let (r, g) be an edge from the path ξ such that $\{r, g\} \cap \{g_3, g_4v\} = \emptyset$. Since $[2,0,1,0] = 4$ there exists a Hamiltonian path $(rv, \theta; gv)$ of $\mathcal{Q}_n^{bot} - \{r_2, g_4\}$. Then by cutting the edge (r, g) and using bridges we can extend one of the paths $(r_3, \mu; g_3)$ or $(r_4, (\nu^R)'v; g_4)$, the one that contains (r, g), with θ and the result will be the desired 2−path covering of $\mathcal{Q}_n - \mathcal{F}$.

B3. $r_3, g_3, g_4 \in \mathcal{Q}_n^{top}; r_4 \in \mathcal{Q}_n^{bot}.$

Let $r^* \neq r_1, r_3$ be a red vertex in $\mathcal{Q}_n^{top}$ not adjacent to g_4, and let $(r_3, \xi; r^*)$ be a Hamiltonian path of $\mathcal{Q}_n^{top} - \{r_1, g_1, g_2\}$. Such Hamiltonian path exists since $[3,1,0,1] = 4$. There are two possibilities: (1) $\xi = \mu\nu\eta$ with $r_3\mu = g_3$ and $g_3\nu = g_4$; and (2) $\xi = \mu\nu\eta$ with $r_3\mu = g_4$ and $g_4\nu = g_3$. In case (1), since $[1,1,1,1] = 4$, there exists a 2−path covering $(g_4(\nu^R)'v, \zeta; r^*(\eta^R)'v)$, $(r_4, \theta; r^*v)$ of $\mathcal{Q}_n^{bot} - \{r_2\}$. The desired 2−path covering of $\mathcal{Q}_n - \mathcal{F}$ is $(r_3, \mu; g_3)$, $(r_4, \theta v(\eta^R)'v\zeta^R v\nu^*; g_4)$. In case (2), since $[1,1,1,1] = 4$, there exists a 2−path covering $(r_3\mu'v, \theta; r^*v)$, $(r_4, \zeta; g_4\nu'v)$ of $\mathcal{Q}_n^{bot} - \{r_2\}$ The desired 2−path covering of $\mathcal{Q}_n - \mathcal{F}$ is $(r_3, \mu'v\theta v\eta^R; g_3), (r_4, \zeta v(\nu')^R; g_4)$.

B4. $r_3, r_4, \in \mathcal{Q}_n^{top}; g_3, g_4 \in \mathcal{Q}_n^{bot}.$

There exists a Hamiltonian path $(r_3, \xi; r_4)$ of $\mathcal{Q}_n^{top} - \{r_1, g_1, g_2\}$, where $\xi = \mu\nu$, with $\{r_3\mu'v, r_3\mu v\} \cap \{r_2, g_3, g_4\} = \emptyset$, for $[3,1,0,1] = 4$. Also, since $[1,1,1,1] = 4$, there exists a 2−path covering $(r_3\mu'v, \eta; g_3), (r_3\mu v, \zeta; g_4)$ of $\mathcal{Q}_n^{bot} - \{r_2\}$. The desired 2−path covering of $\mathcal{Q}_n - \mathcal{F}$ is $(r_3, \mu'v\eta; g_3), (r_4, (\nu^R)v\zeta; g_4)$.

B5. $r_3, g_3 \in \mathcal{Q}_n^{top}; r_4, g_4 \in \mathcal{Q}_n^{bot}.$

Let $r^* \neq r_1, r_3$ be a red vertex in $\mathcal{Q}_n^{top}$ not adjacent to g_3, with $r^*v \neq g_4$. Since $[3,1,0,1] = 4$, there exists a Hamiltonian path $(r_3, \xi; r^*)$ of $\mathcal{Q}_n^{top} - \{r_1, g_1, g_2\}$, where $\xi = \mu\nu$, with $r_3\mu = g_3$. If $r^*(\nu^R)'v = g_4$ then there exists a Hamiltonian path $(r_4, \eta; r^*v)$ of $\mathcal{Q}_n^{bot} - \{r_2, g_4\}$ for $[2,0,1,0] = 4$. Then the desired 2−path covering of $\mathcal{Q}_n - \mathcal{F}$ is $(r_3, \mu; g_3), (r_4, \eta v(\nu^R)'v; g_4)$. If $r^*(\nu^R)'v \neq g_4$ then there exists a 2−path covering $(r_4, \eta; r^*v), (r^*(\nu^R)'v, \zeta; g_4)$ of $\mathcal{Q}_n^{bot} - \{r_2\}$

for $[1, 1, 1, 1] = 4$. The desired $2-$path covering of $\mathcal{Q}_n - \mathcal{F}$ is $(r_3, \mu; g_3)$, $(r_4, \eta v(\nu^R)'v\zeta; g_4)$.

B6. $r_3, g_4 \in \mathcal{Q}_n^{top}$; $r_4, g_3 \in \mathcal{Q}_n^{bot}$.

Let $r^* \neq r_1, r_3$ be a red vertex in $\mathcal{Q}_n^{top}$ with $r^*v \neq g_3$. Since $[3, 1, 0, 1] = 4$, there exists a Hamiltonian path $(r_3, \xi; r^*)$ of $\mathcal{Q}_n^{top} - \{r_1, g_1, g_2\}$, where $\xi = \mu\nu$, with $r_3\mu = g_4$. If $r_3\mu'v = g_3$ then there exists a Hamiltonian path $(r_4, \eta; r^*v)$ of $\mathcal{Q}_n^{bot} - \{r_2, g_3\}$ for $[2, 0, 1, 0] = 4$. Then the desired $2-$path covering of $\mathcal{Q}_n - \mathcal{F}$ is $(r_3, \mu'v; g_3)$, $(r_4, \eta v\nu^R; g_4)$. If $r_3\mu'v \neq g_3$ then there exists a $2-$path covering $(r_3\mu'v, \eta; g_3)$, $(r_4, \zeta; r^*v)$ of $\mathcal{Q}_n^{bot} - \{r_2\}$ for $[1, 1, 1, 1] = 4$. Then the desired $2-$path covering of $\mathcal{Q}_n - \mathcal{F}$ is $(r_3, \mu'v\eta; g_3)$, $(r_4, \zeta v\nu^R; g_4)$.

B7. $r_3 \in \mathcal{Q}_n^{top}$; $g_4, r_4, g_3 \in \mathcal{Q}_n^{bot}$.

Let $r^* \neq r_1$ be a red vertex in $\mathcal{Q}_n^{top}$ with $r^*v \neq g_3, g_4$. Since $[3, 1, 0, 1] = 4$, there exists a Hamiltonian path $(r_3, \xi; r^*)$ of $\mathcal{Q}_n^{top} - \{r_1, g_1, g_2\}$. Also, since $[1, 1, 1, 1] = 4$, there exists a $2-$path covering $(r^*v, \eta; g_3)$, $(r_4, \theta; g_4)$ of $\mathcal{Q}_n^{bot} - \{r_2\}$. The desired $2-$path covering of $\mathcal{Q}_n - \mathcal{F}$ is $(r_3, \xi v\eta; g_3)$, $(r_4, \theta; g_4)$.

B8. $r_3, r_4, g_3, g_4 \in \mathcal{Q}_n^{bot}$.

Without loss of generality we may assume that $g_3v \neq r_1$. Let g be any green vertex in $\mathcal{Q}_n^{bot}$ such that $gv \neq r_1$, and $(r_3, \xi; g)$, $(r_4, \eta; g_4)$ be a $2-$path covering of $\mathcal{Q}_n^{bot} - \{r_2, g_3\}$ ($[2, 0, 2, 0] = 4$). Let also $(gv, \mu; g_3v)$ be a Hamiltonian path of $\mathcal{Q}_n^{top} - \{r_1, g_1, g_2\}$ ($[3, 1, 0, 1] = 4$). The desired $2-$path covering of $\mathcal{Q}_n - \mathcal{F}$ is $(r_3\xi v\mu v; g_3)$, $(r_4, \eta; g_4)$.

B9. $g_3 \in \mathcal{Q}_n^{top}$; $r_3, r_4, g_4 \in \mathcal{Q}_n^{bot}$.

If one of the directions that separates the vertices r_1, r_2 (g_1, g_2) also separates the vertices g_1, g_2 (r_1, r_2) then we fall into **Case A**. This implies that we may assume that $r_2 = r_1xy\xi$, $g_2 = g_1zu\mu$, where x, y, z, u are different letters from **S** and ξ, μ are words free of x, y. We can also assume that neither of the splits in the directions x, y, z, u produces a situation equivalent to one of the subcases **B1** – **B8**. This implies that $r_3 = g_1xy\zeta$, $r_4 = g_1xy\eta$, where ζ, η are words free of x and y, and that $g_3 = r_1zu\theta$, $g_4 = r_1zu\nu$, where θ, ν are words free of z and u. Therefore, without loss of generality, we can assume that $d_H(r_1, g_4) > 1$, hence $g_4v \neq r_1$.

Let $t \in \mathbf{S}$ be such that $t \neq v$ and $g_4vt \notin \{g_1, g_2, g_3\}$. Such t exist for $n \geq 5$. Using $[4] = 4$ ([CG2, Lemma 3.8]) we can find a Hamiltonian cycle (g_4vt, ζ) of $\mathcal{Q}_5^{top} - \{r_1, g_4v, g_1, g_2\}$. Let $\zeta = \theta\gamma$, with $g_4vt\theta = g_3$. Since $[2, 0, 2, 0] = 4$, there exists a $2-$path covering $(g_4vt\theta'v, \mu)$, $(g_3\gamma'v, \eta)$ of $\mathcal{Q}_5^{bot} - \{r_2, g_4\}$. The desired $2-$path covering of $\mathcal{Q}_5 - \mathcal{F}$ is $(r_3\eta^Rv(\gamma')^R; g_3)$, $(r_4, \mu^Rv(\theta')^Rtv; g_4)$.

B10. $g_3, g_4 \in \mathcal{Q}_n^{top}$; $r_3, r_4 \in \mathcal{Q}_n^{bot}$.

If there is a coordinate that separates r_1 from r_2 and g_1 from g_2 then we can choose that coordinate (which leads to **Case A**) and we can avoid the subcase that we are considering. Therefore we can assume that every coordinate that separates r_1 from r_2 does not separate g_1 from g_2. If one of the coordinates that separates r_1 from r_2 separates $g_1, g_2, g_3, g_4, r_3, r_4$ in a way that was considered in subcases **B1** through **B9** then we are done. Thus, we can assume that every coordinate that separates r_1 from r_2 does not separate $g_1, g_2, g_3,$ and g_4 and separates r_3, r_4 from $g_1, g_2, g_3,$ and g_4. Since the situation is symmetrical with respect to green and red vertices we can also assume that every coordinate that separates g_1 from g_2 does not separate $r_1, r_2, r_3,$ and r_4 and separates g_3, g_4 from $r_1, r_2, r_3,$ and r_4. There are at least two coordinates that separate r_1 from r_2. At those two coordinates r_3 and r_4 must coincide, as well as g_1 and g_2. Also, there are at least two coordinates that separate g_1 from g_2. At those two coordinates r_3 and r_4 must also coincide. Therefore r_3 and r_4 must have at least four identical coordinates and since they must differ at least at two coordinates we conclude that this subcase is possible when $n \geq 6$.

Let us choose one coordinate that separates g_1 from g_2. Without loss of generality we can assume that both chosen coordinates separate all vertices $g_1, g_2, g_3, g_4,$ r_1, r_2, r_3, r_4 as follows: $g_2, g_3, g_4 \in \mathcal{Q}_n^{top,bot}$, $r_2, r_3, r_4 \in \mathcal{Q}_n^{bot,top}$ and $g_1, r_1 \in \mathcal{Q}_n^{top,top}$. Clearly, the dimension of each one of these hypercubes is at least four. In the construction of the desired path covering of $\mathcal{Q}_n - \mathcal{F}$ we shall use the letter $v \in \mathbf{S}$ for connecting vertices between the plates that separate r_1 and r_2 and we shall use $u \in \mathbf{S}$ to connect vertices from the plates that separate g_1 from g_2.

Let $x \in \mathbf{S}$ be such that $r_3 x \in \mathcal{Q}_n^{bot,top}$ and $r_3 x v \neq r_1$ and let $y \in \mathbf{S}$ be such that $g_4 y \in \mathcal{Q}_n^{top,bot}$ and $g_4 y u \neq g_1$. Let also $g_5, g_6 \in \mathcal{Q}_n^{bot,top} - \{r_3 x\}$ be such that $g_5 v \neq r_1$ and $r_5, r_6 \in \mathcal{Q}_n^{top,bot} - \{g_4 y\}$ be such that $r_5 u \neq g_1$. Since $[3, 1, 0, 1] = 4$, there exists a Hamiltonian path $(g_5, \xi; g_6)$ of $\mathcal{Q}_n^{bot,top} - \{r_2, r_3, r_3 x\}$ and a Hamiltonian path $(r_5, \zeta; r_6)$ of $\mathcal{Q}_n^{top,bot} - \{g_2, g_4, g_4 y\}$. Let $\xi = \mu\nu$, where $g_5\mu = r_4$ and $\zeta = \eta\theta$, where $r_5\eta = g_3$. Since $[0, 0, 2, 0] = 4$, there exists a 2$-$path covering $(r_5\eta'v, \gamma_1; r_6 v)$, $(g_5\mu'u, \gamma_2; g_6 u)$ of $\mathcal{Q}_n^{bot,bot}$ and since $[2, 0, 2, 0] = 4$ there exists a 2$-$path covering $(g_4 yu, \gamma_3; g_5 v)$, $(r_3 xv, \gamma_4; r_5 u)$ of $\mathcal{Q}_n^{top,top} - \{r_1, g_1\}$. Then $(r_3, xv\gamma_4 u\eta'v\gamma_1 v\theta^R; g_3)$, $(g_4, yu\gamma_3 v\mu'u\gamma_2 u\nu^R; r_4)$ is the desired 2$-$path covering of $\mathcal{Q}_n - \mathcal{F}$. $\qquad\qquad\square$

Now we prove the main result in this paper.

Theorem 3.2 ($[7, 1, 0, 1] = 6$)**.** *Let* $n \geq 6$, r_1, r_2, r_3, r_4, r_5 *be five distinct red vertices, and* g_1, g_2, g_3, g_4 *four distinct green vertices in* $\mathcal{Q}_n$. *Then there exists a Hamiltonian path* $(r_4, \xi; r_5)$ *in* $\mathcal{Q}_n - \mathcal{F}$, *where* $\mathcal{F} = \{r_1, r_2, r_3, g_1, g_2, g_3, g_4\}$. *The claim is not true for* $n = 5$.

Proof. It was proved already in [CG2, Conjecture 6.3] that $[2k, 1, 0, 1] \geq k + 3$. Therefore $[7, 1, 0, 1] \geq 6$.

Let $n \geq 6$. We can assume that $r_1 \in \mathcal{Q}_n^{top}$ and $r_2, r_3 \in \mathcal{Q}_n^{bot}$. There are three cases to consider that depend on the distribution of the red terminals among the plates:

Case A. $r_4, r_5 \in \mathcal{Q}_n^{top}$.

Case B. $r_4 \in \mathcal{Q}_n^{top}$ and $r_5 \in \mathcal{Q}_n^{bot}$.

Case C. $r_4, r_5 \in \mathcal{Q}_n^{bot}$.

For each case we have to consider 5 subcases that we label $\binom{k}{4-k}$, where k is the number of green faulty vertices on the top plate.

Case A. $r_4, r_5 \in \mathcal{Q}_n^{top}$.

Subcase $\binom{4}{0}$. $g_1, g_2, g_3, g_4 \in \mathcal{Q}_n^{top}$.

Let $r^* \neq r_1, r_4, r_5$ be a red vertex in $\mathcal{Q}_n^{top}$ such that $d_H(r^*, g_4) > 2$ and let $(r^*, \xi; r_5)$ be a Hamiltonian path of $\mathcal{Q}_n^{top} - \{r_1, r_4, g_1, g_2, g_3\}$ ($[5, 1, 0, 1] = 5$). Then $\xi = \mu\nu$ with $r^*\mu = g_4$. Let $(r_4v, \eta; r^*v), (r^*\mu'v, \zeta; r_5(\nu^R)'v)$ be a $2-$path covering of $\mathcal{Q}_n^{bot} - \{r_2, r_3\}$ ($[2, 2, 0, 2] = 4$). The desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ is $(r_4, v\eta v\mu'v\zeta v\nu^*; r_5)$.

Subcase $\binom{3}{1}$. $g_1, g_2, g_3 \in \mathcal{Q}_n^{top}$, and $g_4 \in \mathcal{Q}_n^{bot}$.

Either $r_4v \neq g_4$ or $r_5v \neq g_4$. Without loss of generality we may assume that $r_4v \neq g_4$. Let $r^* \neq r_1, r_4, r_5$ be a red vertex in $\mathcal{Q}_n^{top}$ such that $r^*v \neq g_4$ and let $(r^*, \xi; r_5)$ be a Hamiltonian path of $\mathcal{Q}_n^{top} - \{r_1, r_4, g_1, g_2, g_3\}$ ($[5, 1, 0, 1] = 5$). Let also $(r_4v, \eta; r^*v)$ be a Hamiltonian path of $\mathcal{Q}_n^{bot} - \{r_2, r_3, g_4\}$ ($[3, 1, 0, 1] = 4$). The desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ is $(r_4, v\eta v\xi; r_5)$.

Subcase $\binom{2}{2}$. $g_1, g_2 \in \mathcal{Q}_n^{top}$ and $g_3, g_4 \in \mathcal{Q}_n^{bot}$.

Let $(r_4, \xi; r_5)$ be a Hamiltonian path of $\mathcal{Q}_n^{top} - \{r_1, g_1, g_2\}$ ($[3, 1, 0, 1] = 4$). The length of ξ is at least 28. Therefore there exist μ, ν such that $\xi = \mu\nu$ and $\{r_4\mu'v, r_4\mu v\} \cap \{r_2, r_3, g_2, g_3, g_4\} = \emptyset$. Let $(r_4\mu'v, \eta; r_4\mu v)$ be a Hamiltonian path of $\mathcal{Q}_n^{bot} - \{r_2, r_3, g_3, g_4\}$ ($[4, 0, 1, 0] = 5$). The desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ is $(r_4, \mu'v\eta v\nu; r_5)$.

Subcase $\binom{1}{3}$. $g_1 \in \mathcal{Q}_n^{top}$ and $g_2, g_3, g_4 \in \mathcal{Q}_n^{bot}$.

Let g^*, g^{**} be two green vertices in $\mathcal{Q}_n^{top} - \{g_1\}$ such that neither g^*v nor $g^{**}v$ is faulty. Since $[2, 0, 2, 0] = 4$, there exists a $2-$path covering $(r_4, \xi; g^*), (g^{**}, \eta; r_5)$ of $\mathcal{Q}_n^{top} - \{r_1, g_1\}$. Also, since $[5, 1, 0, 1] = 5$, there exists a Hamiltonian path $(g^*v, \zeta; g^{**}v)$ of $\mathcal{Q}_n^{bot} - \{r_2, r_3, g_2, g_3, g_4\}$. The desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ is $(r_4, \xi v\zeta v\eta; r_5)$.

Subcase $\binom{0}{4}$. $g_1, g_2, g_3, g_4 \in \mathcal{Q}_n^{bot}$.

Without loss of generality we can assume that $r_4 v \neq g_1$ and $r_5 v \neq g_2$ by renumbering the deleted green vertices, if necessary. We can find letters $x, y \neq v$ such that $g^* = r_4 x$, $g^{**} = r_5 y$ are two green vertices in $\mathcal{Q}_n^{top}$ such that neither $g^* v$ nor $g^{**} v$ is faulty and $d_H(g^* v, g_1) > 2$, $d_H(g^{**} v, g_2) > 2$. Since $[4, 0, 2, 0] = 5$, there exists a $2-$path covering $(g^* v, \eta; g_1)$, $(g_2, \zeta; g^{**} v)$ of $\mathcal{Q}_n^{bot} - \{r_2, r_3, g_3, g_4\}$. Also, since $[5, 1, 0, 1] = 5$, there exists a Hamiltonian path $(g^* v \eta' v, \mu; g_2 \varphi(\zeta) v)$ of $\mathcal{Q}_n^{top} - \{r_1, r_4, r_5, g^*, g^{**}\}$. The desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ is $(r_4, x v \eta' v \mu v \zeta^* v y; r_5)$.

Case B. $r_4 \in \mathcal{Q}_n^{top}$ and $r_5 \in \mathcal{Q}_n^{bot}$.

Subcase $\binom{4}{0}$. $g_1, g_2, g_3, g_4 \in \mathcal{Q}_n^{top}$.

Let $x \neq v$ be a letter such that $r_5 x v \neq r_1, r_4$ and $d_H(r_5 x v, g_4) > 2$. We choose a red vertex $r^* \neq r_1, r_4$ in $\mathcal{Q}_n^{top}$ such that $d_H(r^*, g_4) > 2$. Let $(r^*, \xi; r_5 x v)$ be a Hamiltonian path of $\mathcal{Q}_n^{top} - \{r_1, r_4, g_1, g_2, g_3\}$ ($[5, 1, 0, 1] = 5$). Then there exists μ, ν such that $\xi = \mu \nu$, with $r^* \mu = g_4$. Let $(r_4 v, \eta; r^* v)$, $(r^* \mu' v, \zeta; r^* \mu \varphi(\nu) v)$ be a $2-$path covering of $\mathcal{Q}_n^{bot} - \{r_2, r_3, r_5, r_5 x\}$ ($[4, 2, 0, 2] = 5$) ($[3, 1, 1, 1] = 5$). The desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ is $(r_4, v \eta v \mu' v \zeta v \nu^* v x; r_5)$.

Subcase $\binom{3}{1}$. $g_1, g_2, g_3 \in \mathcal{Q}_n^{top}$ and $g_4 \in \mathcal{Q}_n^{bot}$.

Let r^*, r^{**}, r^{***} be three distinct red vertices in $\mathcal{Q}_n^{top} - \{r_1, r_4\}$ such that $g_4 \notin \{r^* v, r^{**} v, r^{***} v\}$. There exists a $2-$path covering $(r_4, \xi; r^*)$, $(r^{**}, \eta; r^{***})$ of $\mathcal{Q}_n^{top} - \{r_1, g_1, g_2, g_3\}$ for $[4, 2, 0, 2] = 5$. Let also $(r^* v, \mu; r^{**} v)$, $(r^{***} v, \nu; r_5)$ be a $2-$path covering of $\mathcal{Q}_n^{bot} - \{r_2, r_3, g_4\}$ ($[3, 1, 1, 1] = 5$). The desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ is $(r_4, \xi v \mu v \eta v \nu; r_5)$.

Subcase $\binom{2}{2}$. $g_1, g_2 \in \mathcal{Q}_n^{top}$ and $g_3, g_4 \in \mathcal{Q}_n^{bot}$.

Let r^* be a red vertex in $\mathcal{Q}_n^{top} - \{r_1, r_4\}$, with $r^* v \notin \{g_3, g_4\}$, and $(r_4, \xi; r^*)$ be a Hamiltonian path of $\mathcal{Q}_n^{top} - \{r_1, g_1, g_2\}$ ($[3, 1, 0, 1] = 4$). Let also $(r^* v, \eta; r_5)$ be a Hamiltonian path of $\mathcal{Q}_n^{bot} - \{r_2, r_3, g_3, g_4\}$ ($[4, 0, 1, 0] = 5$). The desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ is $(r_4, \xi v \eta; r_5)$.

Subcase $\binom{1}{3}$. $g_1 \in \mathcal{Q}_n^{top}$ and $g_2, g_3, g_4 \in \mathcal{Q}_n^{bot}$.

Let g^* be a green vertex in $\mathcal{Q}_n^{top} - \{g_1\}$, with $g^* v \notin \{r_2, r_3, r_5\}$, $(r_4, \xi; g^*)$ be a Hamiltonian path of $\mathcal{Q}_n^{top} - \{r_1, g_1\}$ ($[2, 0, 1, 0] = 4$) and $(g^* v, \eta; r_5)$ be a Hamiltonian path of $\mathcal{Q}_n^{bot} - \{r_2, r_3, g_2, g_3, g_4\}$ ($[5, 1, 0, 1] = 5$). The desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ is $(r_4, \xi v \eta; r_5)$.

Subcase $\binom{0}{4}$. $g_1, g_2, g_3, g_4 \in \mathcal{Q}_n^{bot}$.

Let g^* be a green vertex in $\mathcal{Q}_n^{top}$, with $g^* v \notin \{r_2, r_3, r_5\}$ and $d_H(g^* v, g_4) > 2$. Since $[5, 1, 0, 1] = 5$, there exists a Hamiltonian path $(r^* v, \xi; r_5)$ of $\mathcal{Q}_n^{bot} - \{r_2, r_3, g_1, g_2, g_3\}$. Then there exists μ and ν such that $\xi = \mu \nu$, with $g^* v \mu = g_4$. Let $(r_4, \eta; g^*)$, $(g^* v \mu' v, \zeta; r_5 (\nu^R)' v)$ be a $2-$path covering of $\mathcal{Q}_n^{top} - \{r_1\}$ ($[1, 1, 1, 1] = 4$). The desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ is $(r_4, \eta v \mu' v \zeta v \nu^*; r_5)$.

Case C. $r_4, r_5 \in \mathcal{Q}_n^{bot}$.

Subcase $\binom{3}{1}$. $g_1, g_2, g_3 \in \mathcal{Q}_n^{top}$ and $g_4 \in \mathcal{Q}_n^{bot}$.

Let $x \neq v$ be a letter such that r_4x and r_4xv are non-deleted vertices and $g^*, g^{**} \neq g_4$ be two distinct non-deleted green vertices in $\mathcal{Q}_n^{bot}$, with $g^*v \neq r_1$, $g^{**}v \neq r_1$, $d_H(g^*, r_5) > 2$ and $d_H(g^{**}, r_5) > 2$. Let also $(g^*, \xi; g^{**})$ be a Hamiltonian path of $\mathcal{Q}_n^{bot} - \{r_2, r_3, r_4, r_4x, g_4\}$ ($[5, 1, 0, 1] = 5$), where $\xi = \mu\nu$, with $g^*\mu = r_5$. By switching g^* with g^{**}, if necessary, we may assume that $g^*\mu'v \neq r_1$. Let $(r_4xv, \eta; g^*v)$, $(g^*\mu'v, \zeta; g^{**}v)$ be a 2−path covering of $\mathcal{Q}_n^{top} - \{r_1, g_1, g_2, g_3\}$ ($[4, 2, 0, 2] = 5$). The desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ is $(r_4, xv\eta v\mu'v\zeta v\nu^R; r_5)$.

Subcase $\binom{2}{2}$. $g_1, g_2 \in \mathcal{Q}_n^{top}$ and $g_3, g_4 \in \mathcal{Q}_n^{bot}$.

Let g^*, g^{**} be two non-deleted green vertices in $\mathcal{Q}_n^{bot}$ with $g^*v, g^{**}v \neq r_1$. Let also $(g^*v, \xi; g^{**}v)$ be a Hamiltonian path of $\mathcal{Q}_n^{top} - \{r_1, g_1, g_2\}$ ($[3, 1, 0, 1] = 4$) and $(r_4, \eta; g^*)$, $(g^{**}, \zeta; r_5)$ be a 2−path covering of $\mathcal{Q}_n^{bot} - \{r_2, r_3, g_3, g_4\}$ ($[4, 0, 2, 0] = 5$). The desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ is $(r_4, \eta v\xi v\zeta; r_5)$.

Subcase $\binom{1}{3}$. $g_1 \in \mathcal{Q}_n^{top}$ and $g_2, g_3, g_4 \in \mathcal{Q}_n^{bot}$.

There exists a Hamiltonian path $(r_4, \xi; r_5)$ of $\mathcal{Q}_n^{bot} - \{r_2, r_3, g_2, g_3, g_4\}$ since $[5, 1, 0, 1] = 5$. The length of ξ is at least 26, hence there exist words μ, ν such that $\xi = \mu\nu$ and neither $r_4\mu'v$ nor $r_4\mu v$ is deleted. Let $(r_4\mu'v, \eta; r_4\mu v)$ be a Hamiltonian path of $\mathcal{Q}_n^{top} - \{r_1, g_1\}$ ($[2, 0, 1, 0] = 4$). The desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ is $(r_4, \mu'v\eta v\nu; r_5)$.

Subcase $\binom{0}{4}$. $g_1, g_2, g_3, g_4 \in \mathcal{Q}_n^{bot}$.

Since $[4, 0, 2, 0] = 5$, there exists a 2−path covering $(r_4, \xi; g_1)$, $(g_2, \eta; r_5)$ of $\mathcal{Q}_n^{bot} - \{r_2, r_3, g_3, g_4\}$. Also, since $[1, 1, 0, 1] = 2$, there exists a Hamiltonian path $(r_4\xi'v, \zeta; r_5(\eta^R)'v)$ of $\mathcal{Q}_n^{top} - \{r_1\}$. The desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ is $(r_4, \xi'v\zeta v\eta^*; r_5)$.

Subcase $\binom{4}{0}$. $g_1, g_2, g_3, g_4 \in \mathcal{Q}_n^{top}$.

We can assume that all the directions that split the red deleted vertices put one deleted red vertex together with the four deleted green vertices on one plate and the other two red deleted vertices with the two red terminals on the opposite plate. Since there are three deleted red vertices, there are at least three directions that separate them. Hence, all green deleted vertices and both terminals must have at least three identical coordinates.

Let $n = 6$ and let x, y, z, s, t, u be the letters that represent the six directions on which we can split the hypercube. Since all green deleted vertices and the two terminals have at least three identical coordinates we conclude that every two green deleted vertices and the two red terminals cannot differ at mote than two coordinates. Therefore all four deleted green vertices belong to the same three

dimesionl hypercube. Then, without loss of generality, we can assume that only the directions x, y and z separate the vertices r_1, r_2 and r_3 and that

$$g_2 = g_1st, g_3 = g_1su, g_4 = g_1tu, r_4 = g_1xyz, r_5 = g_1xyzst.$$

Let us split $\mathcal{Q}_6$ in the s direction. Then we can assume that g_1, g_4, r_4 are on the top plate and g_2, g_3, r_5 are on the bottom plate. The three red deleted vertices are on one of the two plates and because of the symmetry we can assume that they are on the top plate. Since in $\mathcal{Q}_6^{top}$ there are eleven green vertices not adjacent to r_4, we can choose two green vertices g, g^* in $\mathcal{Q}_6^{top} - \{g_1, g_4\}$ not adjacent to r_4. Then they automatically satisfy $gv, g^*v \neq r_5$. Let $(g, \xi; g^*)$ be a Hamiltonian path of $\mathcal{Q}_6^{top} - \{r_1, r_2, r_3, g_1, g_4\}$ ([5, 1, 0, 1] = 5). Let $\xi = \eta\theta$, with $g\eta = r_4$. We can assume (by reversing the Hamiltonian path, if necessary) that $g\eta'v \neq r_5$. Let $(r_4\theta v, \mu; gv)$, $(g\eta'v, \nu; r_5)$ be a 2−path covering of $\mathcal{Q}_6^{bot} - \{g_2, g_3\}$ ([2, 2, 0, 2] = 4). The desired Hamiltonian path of $\mathcal{Q}_6 - \mathcal{F}$ is $(r_4, \theta v\mu v\eta'v\nu; r_5)$.

This finishes the proof of our claim for $n = 6$.

Let $n \geq 7$ and assume that our claim has been proved for all dimensions between 6 and $n - 1$. We can assume that every direction that splits the two red terminals puts the red deleted vertices on one plate, say the top plate, otherwise we would have a case equivalent to one of the subcases considered above. Moreover, if one of those directions puts two green deleted vertices on the top and two green deleted vertices on the bottom plate then the desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ can be found in the same way as the one in the case $n = 6$ considered above. Therefore we may assume that r_1, r_2, r_3, r_4 are on the top plate and r_5 is on the bottom plate. We need to consider the following subsubcases: (1) g_1, g_2, g_3, g_4 are on the top plate; (2) g_1, g_2, g_3 are on the top plate, g_4 is on the bottom plate; (3) g_1 is on the top plate, g_2, g_3, g_4 are on the bottom plate; and (4) g_1, g_2, g_3, g_4 are on the bottom plate.

Subsubcase (1) Let r be any red vertex in $\mathcal{Q}_n^{top} - \{r_1, r_2, r_3, r_4\}$. By the induction hypothesis there is a Hamiltonian path $(r_4, \xi; r)$ of $\mathcal{Q}_n^{top} - \mathcal{F}$. Let also $(rv, \eta; r_5)$ be any Hamiltonian path of $\mathcal{Q}_n^{bot}$. The desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ is $(r_4, \xi v\eta; r_5)$.

Subsubcase (2) Using [6] = 5 ([CG2, Lemma 4.5]) we can find a Hamiltonian cycle (r_4, ξ) of $\mathcal{Q}_n^{top} - \{r_1, r_2, r_3, g_1, g_2, g_3\}$. We can assume (by reversing the cycle, if necessary) that $r_4\xi'v \neq r_5$. Let $(r_4\xi'v, \eta; r_5)$ be a Hamiltonian path of $\mathcal{Q}_n^{bot} - \{g_4\}$ ([1, 1, 0, 1] = 2). The desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ is $(r_4, \xi'v\eta; r_5)$.

Subsubcase (3) Let g, g^*, g^{**}, g^{***} be four distinct green vertices in the top plate that are different from g_1 and r_5v and are not adjacent to r_4. Since $[4, 2, 0, 2] = 5$, there exists a 2−path covering $(g, \xi; g^*)$, $(g^{**}, \eta; g^{***})$ of $\mathcal{Q}_n^{top} - \{r_1, r_2, r_3, g_1\}$. We can assume that r_4 lies on the path from g to g^* and therefore $\xi = \mu\nu$, with

$g\mu = r_4$. Let $(g^*v, \alpha; gv)$, $(g\mu'v; \beta; g^{**}v)$, $(g^{***}v, \gamma; r_5)$ be a $3-$path covering of $\mathcal{Q}_n^{bot} - \{g_2, g_3, g_4\}$ ($[3, 3, 0, 3] \leq 6$). The desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ is $(r_4, \nu v\alpha v\mu'v\beta v\eta v\gamma; r_5)$.

Subsubcase (4) Assume that this subsubcase cannot be avoided. That means that each direction that splits the green deleted vertices puts the three red deleted vertices on one plate and the two terminals either both on the top plate or both on the bottom plate. We assume that r_1, r_2, r_3 are on the top plate. We have to consider six situations that depend on the number of green deleted vertices on each plate and whether the red terminals are on the top plate or on the bottom plate.

(i) g_1, g_2, g_3, r_4, r_5 are on the top plate, g_4 is on the bottom plate.

Let (r_4, ξ) be a Hamiltonian cycle of $\mathcal{Q}_n^{top} - \{r_1, r_2, r_3, g_1, g_2, g_3\}$ ($[6] = 5$, [CG2, Lemma 4.5]). Then $\xi = \mu\nu$, with $r_4\mu = r_5$. Let $(r_4\mu'v, \eta; r_5\nu'v)$ be any Hamiltonian path of $\mathcal{Q}_n^{bot} - \{g_4\}$ ($[1, 1, 0, 1] = 2$). The desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ is $(r_4, \mu'v\eta v(\nu')^R; r_5)$.

(ii) g_1, g_2, g_3 are on the top plate, g_4, r_4, r_5 are on the bottom plate.

Let g be any non-deleted green vertex on the top plate such that $gv \neq r_4, r_5$ and let (g, ξ) be a Hamiltonian cycle of $\mathcal{Q}_n^{top} - \{r_1, r_2, r_3, g_1, g_2, g_3\}$ ($[6] = 5$ [CG2, Lemma 4.5]). We can assume (by reversing the cycle, if necessary) that $g\xi'v \neq g_4$. Let $(r_4, \eta; gv)$, $(g\xi'v, \zeta; r_5)$ be a $2-$path covering of $\mathcal{Q}_n^{bot} - \{g_4\}$ ($[1, 1, 1, 1] = 4$.) The desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ is $(r_4, \eta v\xi'v\zeta; r_5)$.

(iii) g_1, g_2, r_4, r_5 are on the top plate, g_3, g_4 are on the bottom plate.

Let g, g^* be two green vertices on $\mathcal{Q}_n^{top} - \{g_1, g_2\}$ such that g^* is neither adjacent to g_1 nor to g_2. Let $(g, \xi; g^*)$ be a Hamiltonian path of $\mathcal{Q}_n^{top} - \{r_1, r_2, r_3, g_1, g_2\}$ ($[5, 1, 0, 1] = 5$). We can assume that $\xi = \alpha\beta\gamma$ with $g\alpha = r_4$ and $g\alpha\beta = r_5$. There exists a $2-$path covering $(gv, \eta; g^*(\gamma^R)'v)$, $(g^*v, \zeta; r_5(\beta^R)'v)$ of $\mathcal{Q}_n^{bot} - \{g_3, g_4\}$ for $[2, 2, 0, 2] = 4$. Then the desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ is $(r_4, \alpha^R v\eta v\gamma^* v\zeta v\beta^*; r_5)$.

(iv) g_1, g_2 are on the top plate, g_3, g_4, r_4, r_5 are on the bottom plate.

Let r, r^* be two red vertices in $\mathcal{Q}_n^{bot} - \{r_4, r_5\}$ such that $\{rv, r^*v\} \cap \{g_1, g_2\} = \emptyset$. There exists a $2-$path covering $(r_4, \xi; r)$, $(r^*, \eta; r_5)$ of $\mathcal{Q}_n^{bot} - \{g_3, g_4\}$ for $[2, 2, 0, 2] = 4$. Also, there exists a Hamiltonian path $(rv, \zeta; r^*v)$ of $\mathcal{Q}_n^{top} - \{r_1, r_2, r_3, g_1, g_2\}$ ($[5, 1, 0, 1] = 5$). The desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ is $(r_4, \xi v\zeta v\eta; r_5)$.

(v) g_1, r_4, r_5 are on the top plate, g_2, g_3, g_4 are on the bottom plate.

Let $g = r_4x, g^* = r_5y$ be two green vertices in $\mathcal{Q}_n^{top} - \{g_1\}$. Moreover, if r_4, r_5 have common neighbors on $\mathcal{Q}_n^{top}$ we select x and y such that the common neighbors of r_4, r_5 are contained in the set $\{g_1, g, g^*\}$. Let (r_4, ξ) be a Hamiltonian cycle in $\mathcal{Q}_n^{top} - \{r_1, r_2, r_3, g_1, g, g^*\}$ ($[6] = 5$ [CG2, Lemma 4.5]). Then $\xi = \eta\theta$,

with $r_4\eta = r_5$. Observe that by the choice of x, y we have $|\eta|, |\theta| > 2$. Let $(gv, \alpha; r_5(\eta^R)'v)$, $(r_4\eta'v, \beta; r_4(\theta^R)'v)$, $(r_5\theta'v, \gamma; g^*v)$ be a $3-$path covering of $\mathcal{Q}_n^{bot} - \{g_2, g_3, g_4\}$ ($[3, 3, 0, 3] \leq 6$). The desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ is $(r_4, xv\alpha v(\eta^*)'v\beta v(\theta^*)'v\gamma vy; r_5)$.

(vi) g_1 is on the top plate, g_2, g_3, g_4, r_4, r_5 are on the bottom plate.

We can assume (by renumbering the terminals, if necessary) that $r_4v \neq g_1$. Let g, g^*, g^{**} be three green vertices in $\mathcal{Q}_n^{top} - \{g_1, r_4v, r_5v\}$ and let $(r_4v, \xi; g)$, $(g^*, \eta g^{**})$ be a $2-$path covering of $\mathcal{Q}_n^{top} - \{r_1, r_2, r_3, g_1\}$ ($[4, 2, 0, 2] = 5$). Let also $(gv, \mu; g^*v)$, $(g^{**}v, \nu; r_5)$ be a $2-$path covering of $\mathcal{Q}_n^{bot} - \{g_2, g_3, g_4, r_4\}$ ($[4, 2, 0, 2] = 5$). The desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ is $(r_4v\xi v\mu v\eta v\nu; r_5)$.

$\square$

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