ON k-SEMI-PERFECT 1-FACTORIZATIONS OF $\mathcal{Q}_n$ AND CRAFT'S CONJECTURE

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Abstract. Let $n \geq 2$, $\mathcal{G}$ be an n-regular graph and $1 \leq k \leq n-1$. An 1-factorization of the graph $\mathcal{G}$ into 1-factors $F_1, \ldots, F_n$ shall be called *k-semi-perfect*, if $F_i \cup F_j$ forms a Hamiltonian cycle for every $1 \leq i \leq k$ and every $k+1 \leq j \leq n$.

The following results about the binary hypercube $\mathcal{Q}_n$ are proved in this paper:

Theorem 1. (a) If $k = 1$ or $k = 5$ and $p \geq 1$ then there exists a k-semi-perfect 1-factorization of $\mathcal{Q}_{k+2p}$.

(b) If $k \geq 1$ and $p \geq 1$ then there exists a $2k$-semi-perfect 1-factorization of $\mathcal{Q}_{2k+2p}$.

Theorem 2. Let $n \geq 3$ and $\mathcal{F}$ be a set of edges in $\mathcal{Q}_n$. Assume also that either

(a) n is odd and $0 \leq |\mathcal{F}| \leq n-2$; or

(b) n is even and $1 \leq |\mathcal{F}| \leq n-2$.

Then there exist at least $k = n - |\mathcal{F}| - 1$ Hamiltonian cycles in $\mathcal{Q}_n - \mathcal{F}$ that intersect on a perfect matching.

Also, a solution of Craft's conjecture for the case $n = 6$ is provided and a conjecture that implies Craft's conjecture for every $n \geq 5$ is formulated.

1. Introduction

Let $\mathcal{G}$ be a simple graph. $\mathcal{G}$ is *decomposable into Hamiltonian cycles* if there exist edge-disjoint Hamiltonian cycles which union covers all edges of $\mathcal{G}$. A set M of edges of $\mathcal{G}$ is called *matching* if every vertex of $\mathcal{G}$ is incident with at most one edge of M. A vertex v of $\mathcal{G}$ is *covered by M* if v is incident with an edge of M. A matching M is called *perfect* if every vertex of $\mathcal{G}$ is covered by M. Perfect matchings are also called 1-*factors*. A *proper* edge coloring of $\mathcal{G}$ is an edge coloring for which every two edges with a common vertex have different colors. Clearly, if $\mathcal{G}$ is *n-regular* (every vertex is incident to exactly n edges) and properly colored with n different colors, then every *color class*,

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i.e. all edges colored in one fixed color, is a perfect matching of $\mathcal{G}$, or equivalently, 1-factor. A proper edge coloring of an n-regular graph $\mathcal{G}$ with n colors is also called 1-*factorization*. An 1-factorization is called *perfect* if the union of any two 1-factors (color classes, perfect matchings) is a Hamiltonian cycle. An 1-factorization of $\mathcal{G}$ into 1-factors $F_1, \dots, F_n$ is called *semi-perfect* if $F_1 \cup F_i$ forms a Hamiltonian cycle for any $1 < i \leq n$.

We extend the definition of semi-perfect 1-factorization as follows.

Definition 1.1. *Let $n \geq 2$, $\mathcal{G}$ be an n-regular graph and $1 \leq k \leq n-1$. An 1-factorization of the graph $\mathcal{G}$ into 1-factors $F_1, \dots, F_n$ shall be called k-*semi-perfect*, if $F_i \cup F_j$ forms a Hamiltonian cycle for every $1 \leq i \leq k$ and every $k+1 \leq j \leq n$.*

It is clear from the definition that an 1-factorization is k-semi-perfect if and only if it is $(n-k)$-semi-perfect. Also, in our terminology, the 1-semi-perfect 1-factorizations (or, equivalently, the $(n-1)$-semi-perfect 1-factorizations) are the well-known semi-perfect 1-factorizations.

The question of existence of perfect or semi-perfect 1-factorizations for different graphs $\mathcal{G}$ has been extensively studied (see [MR], [P], [K]). In this paper we study the existence of k-semi-perfect 1-factorizations for the n-regular graph $\mathcal{Q}_n$, where $n \geq 2$. We recall that the $n-$dimensional binary hypercube $\mathcal{Q}_n$ is the graph whose vertices are the binary sequences of length n and whose edges are pairs of binary sequences that differ in exactly one position. All edges that connect vertices that differ at a given position i, $1 \leq i \leq n$, are called *parallel* and define a *direction* that we denote also by i.

2. On Craft's conjecture

In 1995 David Craft formulated the following conjecture (see [C]):

Conjecture 2.1 (D. Craft). *For each integer $n \geq 2$ there is a semi-perfect 1-factorization of $\mathcal{Q}_n$.*

The following theorem settles Craft's conjecture for odd n. [1]

Theorem 2.2. *If $n \geq 3$ is odd then there exists 1-semi-perfect 1-factorization of $\mathcal{Q}_n$.*

Proof. In what follows we shall use the following well-known fact: For any $p \geq 1$ the hypercube $\mathcal{Q}_{2p}$ is decomposable into p Hamiltonian cycles ([ABS], [B]).

[1]We are grateful to Dan Archdeacon for helpful e-mail correspondence and to Giuseppe Mazzuoccolo who informed us, while we were writing this paper, that a proof of Craft's conjecture for the case when n is odd appeared already in [KK].

Let $n = 2p + 1 \geq 3$. It is convenient to view the hypercube $\mathcal{Q}_{2p+1}$ as two copies $\mathcal{Q}^0_{2p}$ and $\mathcal{Q}^1_{2p}$ of the $2p-$dimensional hypercube $\mathcal{Q}_{2p}$ such that every vertex v^0 in $\mathcal{Q}^0_{2p}$ is connected by an edge to the vertex v^1 in $\mathcal{Q}^1_{2p}$ which is the same binary sequence of length $2p$ as v^0. Clearly, the set M of all edges from $\mathcal{Q}_n$ that connect vertices from $\mathcal{Q}^0_{2p}$ and $\mathcal{Q}^1_{2p}$ form an 1-factor of $\mathcal{Q}_n$.

Let $H_1, H_2, \ldots, H_p$ be a decomposition of $\mathcal{Q}_{2p}$ into p Hamiltonian cycles and let $H^i_1, H^i_2, \ldots, H^i_p$ be the corresponding Hamiltonian decomposition of $\mathcal{Q}^i_{2p}$, $i \in \{0, 1\}$. Every Hamiltonian cycle H_j, $1 \leq j \leq p$ defines in a natural way two 1-factors $H_j(o)$ and $H_j(e)$ in $\mathcal{Q}_{2p}$. (Starting from an edge, follow the cycle H_j (which has an even number of edges) and enumerate all edges. Then take for $H_j(o)$ all odd-numbered edges and for $H_j(e)$ all even-numbered edges.) Let $H^i_1(x), H^i_2(x), \ldots, H^i_p(x)$, $x \in \{o, e\}$, $i \in \{0, 1\}$, be the corresponding sets of 1-factors of $\mathcal{Q}^i_{2p}$, $i \in \{0, 1\}$ defined by $H_1, H_2, \ldots, H_p$. Then

$$\{M, H^0_1(o) \cup H^1_1(e), H^0_1(e) \cup H^1_0(o), \ldots, H^0_p(o) \cup H^1_p(e), H^0_p(e) \cup H^1_p(o)\}$$

form the required 1-semi-perfect 1-factorization of $\mathcal{Q}_n$ for the union of M and any other 1-factor forms a Hamiltonian cycle. □

Semi-perfect 1-factorization of $\mathcal{Q}_3$ and $\mathcal{Q}_4$ were known to D. Craft. Using a computer program we were able to find all semi-perfect 1-factorizations of $\mathcal{Q}_4$ and many different semi-perfect 1-factorizations of $\mathcal{Q}_6$. One of those semi-perfect 1-factorizations of $\mathcal{Q}_6$ is given in Appendix A, where each F_i, $1 \leq i \leq 6$, is a perfect matching of $\mathcal{Q}_6$ and $F_1 \cup F_i$, $2 \leq i \leq 6$, is a Hamiltonian cycle of $\mathcal{Q}_6$. Since all edges in F_1 are parallel to one direction, it easily follows that the projections parallel to that direction of all edges from F_i, $2 \leq i \leq 6$, onto $\mathcal{Q}_5$, form a Hamiltonian cycle of $\mathcal{Q}_5$. Therefore there exist two partitions of the set of edges of $\mathcal{Q}_5$ into perfect matchings $\{M_1, M_2, \ldots, M_5\}$ and $\{N_1, N_2, \ldots, N_5\}$ such that $M_i \cup N_i$ forms a Hamiltonian cycle for each $1 \leq i \leq 5$. It is interesting to note that a semi-perfect 1-factorization of $\mathcal{Q}_4$ with a perfect matching, all edges of which are parallel to one direction, does not exist; or equivalently, two partitions of the set of edges of $\mathcal{Q}_3$ into perfect matchings $\{M_1, M_2, M_3\}$ and $\{N_1, N_2, N_3\}$ such that $M_i \cup N_i$ forms a Hamiltonian cycle for each $1 \leq i \leq 3$, do not exist.

Since for any $p \geq 1$ the hypercube $\mathcal{Q}_{2p}$ is decomposable into p Hamiltonian cycles, it easily follows that for every even $n \geq 2$ there exist two partitions of the set of edges of $\mathcal{Q}_n$ into perfect matchings $\{M_1, M_2, \ldots, M_n\}$ and $\{N_1, N_2, \ldots, N_n\}$ such that $M_i \cup N_i$ forms a

Hamiltonian cycle for each $1 \le i \le n$. Based on the above observations we state the following conjecture.

Conjecture 2.3. *Let $n = 2$ or $n \ge 4$. Then there exist two 1-factorizations $\{M_1, M_2, \dots, M_n\}$ and $\{N_1, N_2, \dots, N_n\}$ of $\mathcal{Q}_n$ such that $M_i \cup N_i$ forms a Hamiltonian cycle for each $1 \le i \le n$.*

We emphasize again that the above conjecture is a theorem for every even positive integer n and for $n = 5$, but it is not true for $n = 3$. Also, Craft's conjecture (for $n \ge 5$) is an easy corollary from it.

3. On k-semi-perfect 1-factorizations of $\mathcal{Q}_n$

Another corollary of Conjecture 2.3 is the following.

Theorem 3.1. *Let $k \ge 2$, $p \ge 1$ be two integers and let Conjecture 2.3 hold true for k. Then there exists a k-semi-perfect 1-factorization of $\mathcal{Q}_{k+2p}$.*

Proof. It is convenient to view the $n-$dimensional hypercube $\mathcal{Q}_{k+2p}$ as a $2p-$dimensional hypercube $\mathcal{Q}_{2p}$ which "vertices" are $k-$dimensional hypercubes $\mathcal{Q}_k$ (i.e. we view $\mathcal{Q}_{k+2p}$ as a Cartesian product of $\mathcal{Q}_{2p}$ and $\mathcal{Q}_k$). We enumerate all vertices of $\mathcal{Q}_{2p}$ from 1 to 2^{2p} (we take the binary representation of each vertex plus one) and we denote by $\mathcal{Q}_k^i$, $1 \le i \le 2^{2p}$, the $k-$dimensional hypercube which is in the "vertex" of $\mathcal{Q}_{2p}$ numbered i. Also, we enumerate all vertices of $\mathcal{Q}_k$ from 1 to 2^k (the binary representation of each vertex plus one) and for each $1 \le i \le 2^{2p}$ we take that same enumeration of the vertices in $\mathcal{Q}_k^i$.

Let $H_1, H_2, \dots, H_p$ be a decomposition of $\mathcal{Q}_{2p}$ into p Hamiltonian cycles. For convenience, we enumerate the edges of each such Hamiltonian cycle starting from an edge that contains the vertex numbered 1. Then every Hamiltonian cycle H_j, $1 \le j \le p$ defines in a natural way two 1-factors $H_j(o)$ and $H_j(e)$ in $\mathcal{Q}_{2p}$. Let $\{H_1(x), H_2(x), \dots, H_p(x)\}$, $x \in \{o, e\}$, be the corresponding two sets of 1-factors of $\mathcal{Q}_{2p}$ defined by $H_1, H_2, \dots, H_p$.

Let also $\{L_1(x), L_2(x), \dots, L_k(x)\}$, $x \in \{o, e\}$ be two 1-factorizations of $\mathcal{Q}_k$ such that $L_r := L_r(e) \cup L_r(o)$ is a Hamiltonian cycle for each $1 \le r \le k$ and let $\{L_1^i(x), L_2^i(x), \dots, L_k^i(x)\}$, $x \in \{o, e\}$, be the corresponding two sets of 1-factors of $\mathcal{Q}_k^i$, $1 \le i \le 2^{2p}$, defined by $L_1(x), L_2(x), \dots, L_k(x)$.

We define the first set of k 1-factors of $\mathcal{Q}_{k+2p}$ in the following way:

$$M_r := L_r^1(e) \cup \bigcup_{2 \le i \le 2^{2p}} L_r^i(o),\ 1 \le r \le k.$$

Since $\mathcal{Q}_{2p}$ is a bipartite graph, and according to our enumeration of the edges, each edge in $H_t(e)$, following the cycle H_t, connects even numbered vertices i_1 in $\mathcal{Q}_{2p}$ (that could be considered as vertices in $\mathcal{Q}_k^{i_1}$) to odd numbered vertices i_2 in $\mathcal{Q}_{2p}$ (that could be considered as vertices in $\mathcal{Q}_k^{i_2}$), and that each edge in $H_t(o)$, following the cycle H_t, connects odd numbered vertices i_1 in $\mathcal{Q}_{2p}$ to even numbered vertices i_2 in $\mathcal{Q}_{2p}$. Notice also that for each j, $1 \le j \le p$, each edge in $H_j(e)$ and $H_j(o)$ corresponds to 2^k edges in $\mathcal{Q}_n$. Let $H_j(x,y)$ be the set of all edges in $\mathcal{Q}_n$ corresponding to such edges from $H_j(x)$ that, following the cycle H_j, begin from a vertex with parity y in $\mathcal{Q}_k^i$ whenever i has parity x, where $1 \le i \le 2^{2p}$.

Now we define the second set of $2p$ 1-factors in the following way:

$$N_j(y) := H_j(e,y) \cup H_j(o,y),\ 1 \le j \le p,\ y \in \{e,o\}.$$

It follows from the definitions that

$$\mathcal{F} := \bigcup\{M_r : 1 \le r \le k\} \cup \bigcup\{\{N_j(e), N_j(o)\} : 1 \le j \le p\}$$

is a set of $k+2p$ pairwise disjoint 1-factors. To finish the proof we need to show that $\mathcal{F}$ is a k-semi-perfect 1-factorization for $\mathcal{Q}_{k+2p}$, i.e. that $M_r \cup N_j(y)$ is a Hamiltonian cycle for each $1 \le r \le k$, each $y \in \{e,o\}$ and each $1 \le j \le p$.

We fix $1 \le r \le k$, $y \in \{e,o\}$ and $1 \le j \le p$. We shall show that $C := M_r \cup N_j(y)$ is a Hamiltonian cycle.

Below if v is a vertex in $\mathcal{Q}_k$ numbered s then the corresponding vertex in $\mathcal{Q}_k^i$ is denoted by $s(i)$.

Let $1 = s_1, s_2, \ldots, s_{2^k}$ be the sequence of numbers of the vertices in $\mathcal{Q}_k$ beginning from 1 that represents the cycle L_r. We can arrange (by changing the direction, if necessary) that the edge $(s_1(1), s_2(1))$ does not belong to C. Let $1 = q_1, q_2, \ldots, q_{2^{2p}}$ be a sequence of numbers of the vertices in $\mathcal{Q}_{2p}$ beginning from 1 that represents the cycle H_j. If the edge $(s_1(1), s_1(q_2))$ belongs to C, then the following vertices belong to C and the edges between them form a path:

$$s_1(1), s_1(q_2), s_2(q_2), s_2(q_3), s_1(q_3), \ldots, s_1(q_{2^{2p}}), s_2(q_{2^{2p}}), s_2(1), s_3(1).$$

The length of this path is 2^{2p+1} and it contains all vertices of the type $s_1(q_i)$ and $s_2(q_i)$, where $1 \le i \le 2^{2p}$, and connects $s_1(1)$ with $s_3(1)$. Using similar paths we can connect $s_3(1)$ with $s_5(1)$, and so on, $s_{2^k-1}(1)$ with $s_1(1)$. There are 2^{k-1} such paths which are edge disjoint. All these paths form a Hamiltonian cycle of $\mathcal{Q}_{k+2p}$ that coincides with C.

In a similar way, if the edge $(s_1(1), s_1(q_2))$ does not belong to C, then $(s_2(1), s_2(q_2))$ belongs to C. Then the following vertices belong to C and the edges between them form a path:

$$s_2(1), s_2(q_2), s_1(q_2), s_1(q_3), s_2(q_3), \ldots, s_2(q_{2^{2p}}), s_1(q_{2^{2p}}), s_1(1), s_{2^k}(1).$$

The length of this path is 2^{2p+1} and it contains all vertices of the type $s_1(q_i)$ and $s_2(q_i)$, where $1 \leq i \leq 2^{2p}$, and connects $s_2(1)$ with $s_{2^k}(1)$. Using similar paths we can connect $s_{2^k}(1)$ with $s_{2^k-1}(1)$, and so on, $s_4(1)$ with $s_2(1)$. There are 2^{k-1} such paths which are edge disjoint. All these paths form a Hamiltonian cycle of $\mathcal{Q}_{k+2p}$ that coincides with C. □

As a corollary of the above theorem we obtain the following.

Corollary 3.2. *(a) If $n \geq 4$ is even then for every even k, $2 \leq k \leq n-2$, there exists a k-semi-perfect 1-factorization of $\mathcal{Q}_n$.*
(b) If $p \geq 1$ then there exists a 5-semi-perfect 1-factorization of $\mathcal{Q}_{5+2p}$.

Using (a) of the above corollary, Theorem 2.2 and some observations from the proof of Theorem 3.1 we prove the following.

Theorem 3.3. *Let $n \geq 3$ and $\mathcal{F}$ be a set of edges in $\mathcal{Q}_n$. Assume also that either*
(a) n is odd and $0 \leq |\mathcal{F}| \leq n-2$; or
(b) n is even and $1 \leq |\mathcal{F}| \leq n-2$.
Then there exist at least $k = n - |\mathcal{F}| - 1$ Hamiltonian cycles in $\mathcal{Q}_n - \mathcal{F}$ that intersect on a perfect matching.

Proof. (a) Let $n \geq 3$ be odd. Then, according to Theorem 2.2, there exists 1-semi-perfect 1-factorization of $\mathcal{Q}_n$. From its proof we know that one of the 1-factors (say M) could be chosen such that all of its edges to be parallel to a chosen direction. We choose a direction in which there are no deleted edges (clearly such a direction exists). Since there are only $|\mathcal{F}|$ deleted edges, at least $k = n - |\mathcal{F}| - 1 \geq 1$ of the remaining 1-factors do not contain deleted edges. Each one of these 1-factors together with M forms a Hamiltonian cycle of $\mathcal{Q}_n - \mathcal{F}$. Therefore there are at least k Hamiltonian cycles of $\mathcal{Q}_n - \mathcal{F}$ that intersect on the perfect matching M.

(b) Let $n \geq 4$ be even. Since $1 \leq |\mathcal{F}| \leq n-2$ there exists at least one direction i_1 such that there is an edge in $\mathcal{F}$ parallel to that direction. Also, there exists a direction i_2 such that no edge in $\mathcal{F}$ is parallel to that direction. Now, as in the proof of Theorem 3.1, we view the $n-$dimensional hypercube $\mathcal{Q}_n$ as an $n-2-$dimensional hypercube $\mathcal{Q}_{n-2}$ which "vertices" are $2-$dimensional hypercubes $\mathcal{Q}_2$ (i.e. we view $\mathcal{Q}_n$ as a Cartesian product of $\mathcal{Q}_{n-2}$ and $\mathcal{Q}_2$). Clearly, we can arrange i_1 and i_2 to belong to $\mathcal{Q}_2$. Then, according to Theorem 3.1, there exists a 2-semi-perfect 1-factorization of $\mathcal{Q}_n$. Since $2^{n-2} > n-2$ there exists a vertex in $\mathcal{Q}_{n-2}$ such that no edge from $\mathcal{F}$ belongs to its "vertex" $\mathcal{Q}_2$.

Therefore, if in the proof of Theorem 3.1 we begin the enumeration of the vertices of $\mathcal{Q}_{n-2}$ from that particular vertex then one of the two 1-factors

$$M(x) := L^1(\overline{x}) \cup \bigcup_{2\leq i\leq 2^{n-2}} L^i(x),\ x \in \{e,o\}.$$

will not contain edges from $\mathcal{F}$. Denote that 1-factor by M. Since there are only $|\mathcal{F}|$ deleted edges, at least $k = n - |\mathcal{F}| - 1 \geq 1$ of the $n-2$ 1-factors defined in the proof of Theorem 3.1 by

$$N_j(y) := H_j(e,y) \cup H_j(o,y),\ 1 \leq j \leq \tfrac{n-2}{2},\ y \in \{e,o\}$$

do not contain deleted edges. Each one of these 1-factors together with M forms a Hamiltonian cycle of $\mathcal{Q}_n - \mathcal{F}$. Therefore there are at least k Hamiltonian cycles of $\mathcal{Q}_n - \mathcal{F}$ that intersect on the perfect matching M. □

As a direct corollary of Theorem 3.3 we obtain the following result (see [LZB] and [SSB]).

Corollary 3.4. *Let $n \geq 2$ and $\mathcal{F}$ be a set of up to $n-2$ edges in $\mathcal{Q}_n$. Then $\mathcal{Q}_n - \mathcal{F}$ is Hamiltonian.*

Appendix A. Solution of Craft's conjecture for $n = 6$

Below we provide one of the solutions of Craft's conjecture for $n = 6$. Since $F_1, \ldots, F_6$ form an 1-factorization of $\mathcal{Q}_6$, the enumeration of the vertices of $\mathcal{Q}_6$ from 1 to 64 should be clear. The 1-factor that forms a Hamiltonian cycle with any other 1-factor is F_1.

$$F_1 = \{(1,17),(2,18),(3,19),(4,20),(5,21),(6,22),(7,23),(8,24),\\ (9,25),(10,26),(11,27),(12,28),(13,29),(14,30),(15,31),(16,32),\\ (33,49),(34,50),(35,51),(36,52),(37,53),(38,54),(39,55),(40,56),\\ (41,57),(42,58),(43,59),(44,60),(45,61),(46,62),(47,63),(48,64)\};$$

$$F_2 = \{(1,2),(3,4),(5,6),(7,8),(9,10),(11,12),(13,14),(15,16),\\ (17,19),(18,24),(20,22),(21,29),(23,39),(25,27),(26,32),(28,30),\\ (31,47),(33,35),(34,42),(36,44),(37,45),(38,40),(41,43),(46,48),\\ (49,50),(51,53),(52,54),(55,56),(57,58),(59,60),(61,62),(63,64)\};$$

$$F_3 = \{(1,3),(2,4),(5,7),(6,8),(9,11),(10,12),(13,15),(14,16),\\ (17,18),(19,21),(20,36),(22,38),(23,24),(25,31),(26,42),(27,28),\\ (29,30),(32,48),(33,41),(34,40),(35,37),(39,47),(43,44),(45,46),\\ (49,51),(50,52),(53,55),(54,62),(56,64),(57,59),(58,60),(61,63)\};$$

$$F_4 = \{(1,7),(2,8),(3,5),(4,6),(9,15),(10,16),(11,13),(12,14),\\ (17,33),(18,20),(19,35),(21,22),(23,31),(24,40),(25,26),(27,43),\\ (28,44),(29,45),(30,32),(34,36),(37,39),(38,46),(41,42),(47,48),\\ (49,55),(50,58),(51,59),(52,60),(53,61),(54,56),(57,63),(62,64)\};$$

$$F_5 = \{(1,9),(2,10),(3,11),(4,12),(5,13),(6,14),(7,15),(8,16),\\ (17,23),(18,34),(19,20),(21,37),(22,24),(25,41),(26,28),(27,29),\\ (30,46),(31,32),(33,39),(35,43),(36,38),(40,48),(42,44),(45,47),\\ (49,57),(50,56),(51,52),(53,54),(55,63),(58,64),(59,61),(60,62)\};$$

$$F_6 = \{(1,49),(2,50),(3,51),(4,52),(5,53),(6,54),(7,55),(8,56),\\ (9,57),(10,58),(11,59),(12,60),(13,61),(14,62),(15,63),(16,64),\\ (17,25),(18,26),(19,27),(20,28),(21,23),(22,30),(24,32),(29,31),\\ (33,34),(35,36),(37,38),(39,40),(41,47),(42,48),(43,45),(44,46)\}.$$

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