ON PATH COVERINGS OF HYPERCUBES WITH ONE FAULTY VERTEX

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ABSTRACT. Let $n \geq 5$ and $\mathcal{Q}_n$ be the $n-$dimensional binary hypercube. Also, let u be a faulty even (odd) vertex in $\mathcal{Q}_n$, u_1 and u_2 be distinct even (odd) vertices of $\mathcal{Q}_n - \{u\}$ and v_1, v_2, v_3 and v_4 be distinct odd (even) vertices of $\mathcal{Q}_n - \{u\}$. We prove that there exist three paths in $\mathcal{Q}_n - \{u\}$, one joining u_1 and u_2, one joining v_1 and v_2 and one joining v_3 and v_4, such that every vertex of $\mathcal{Q}_n - \{u\}$ belongs to exactly one of the paths. Also, under the same hypotheses, there exist three paths in $\mathcal{Q}_n - \{u\}$, one joining u_1 and v_1, one joining u_2 and v_2 and one joining v_3 and v_4, such that every vertex of $\mathcal{Q}_n - \{u\}$ belongs to exactly one of the paths.

1. Introduction

A *path covering* of a graph $G = (V, E)$ is a set of vertex-disjoint simple paths in G with the property that every vertex of G is contained in exactly one of the paths. For $k \geq 1$, a $k-$path covering of G is a path covering consisting of k paths. In this paper we are interested in $3-$path coverings of induced subgraphs of the $n-$dimensional binary hypercube $\mathcal{Q}_n$. An induced subgraph of $\mathcal{Q}_n$ is also known as a *faulty hypercube.*

The vertices of $\mathcal{Q}_n$ are binary vectors of length n. A vertex is *odd* if it contains an odd number of 1s and *even* if it contains an even number of 1s. In order to make our results as general as possible, the vertices of one parity will be called *green* and those of the other parity will be called *red.*

One of the most basic theorems on path coverings of $\mathcal{Q}_n$ is Havel's Lemma [1], which states that given any two vertices of opposite parity in $\mathcal{Q}_n$, with $n \geq 1$, there exists a Hamiltonian path with these two vertices as endpoints, or in the terminology of [2], $\mathcal{Q}_n$, for each $n \geq 1$,

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is Hamilton laceable. For $2-$path coverings of $\mathcal{Q}_n$ the following two results are known.

Lemma 1.1 ([3]). *Let $n \geq 2$, $g_1, g_2 \in V(\mathcal{Q}_n)$ be distinct green vertices and $r_1, r_2 \in V(\mathcal{Q}_n)$ be distinct red vertices. Then there exists a $2-$path covering of $\mathcal{Q}_n$ with one path joining g_1 and r_1 and another path joining g_2 and r_2.*

Lemma 1.2 ([4], [5]). *Let $n \geq 4$, $g_1, g_2 \in V(\mathcal{Q}_n)$ be distinct green vertices and $r_1, r_2 \in V(\mathcal{Q}_n)$ be distinct red vertices. Then there exists a $2-$path covering of $\mathcal{Q}_n$ with one path joining g_1 and g_2 and another joining r_1 and r_2. The claim is not true when $2 \leq n \leq 3$.*

For $3-$path coverings of $\mathcal{Q}_n$ the following result was obtained in [5].

Lemma 1.3 ([5]). *Let $n \geq 5$, $g_1, g_2, g_3 \in V(\mathcal{Q}_n)$ be distinct green vertices and $r_1, r_2, r_3 \in V(\mathcal{Q}_n)$ be distinct red vertices. Then there exists a $3-$path covering of $\mathcal{Q}_n$ with one path joining g_1 and r_1, another path joining g_2 and r_2, and another path joining g_3 and r_3. The claim is not true when $3 \leq n \leq 4$.*

Another basic result on path coverings of $\mathcal{Q}_n$ with one deleted vertex is the following.

Lemma 1.4 ([6]). *Let $n \geq 2$, $g \in V(\mathcal{Q}_n)$, and $r_1, r_2 \in V(\mathcal{Q}_n)$ be distinct red vertices. Then there exist a Hamiltonian path for $\mathcal{Q}_n - \{g\}$ joining g_1 and g_2.*

In the terminology of [6] the above lemma states that $\mathcal{Q}_n$ is hyper-Hamilton laceable.

For $2-$path coverings of $\mathcal{Q}_n$ with one deleted vertex the following result is proved in [5].

Lemma 1.5 ([5]). *Let $n \geq 4$, $g, g_1 \in V(\mathcal{Q}_n)$ be distinct green vertices and $r_1, r_2, r_3 \in V(\mathcal{Q}_n)$ be distinct red vertices. Then there exists a $2-$path covering of $\mathcal{Q}_n - \{g\}$ with one path joining g_1 and r_1 and another joining r_2 and r_3. The claim is not true when $2 \leq n \leq 3$.*

Castañeda and Gotchev continued the study of path coverings of induced subgraphs of $\mathcal{Q}_n$ in [5] where they considered the following more general problem:

Problem 1.6. *Let M, C, N and O be nonnegative integers and $\mathcal{F}$ be a subset of $V(\mathcal{Q}_n)$ such that the cardinality of $\mathcal{F}$ is M and the absolute value of the difference between the number of odd vertices in $\mathcal{F}$ and the number of even vertices in $\mathcal{F}$ is C. Let also $a_1, \ldots, a_{N+O}$ and*

$b_1, \dots, b_{N+O}$ be distinct vertices in $V(\mathcal{Q}_n) \setminus \mathcal{F}$, such that there are N pairs a_i, b_i of opposite parities and O pairs a_i, b_i of matching parities. Find the smallest m such that if $n \geq m$ then, regardless of the choice of $\mathcal{F}$ and $a_1, \dots, a_{N+O}, b_1, \dots, b_{N+O}$, there exists an $(N+O)-$path covering of the subgraph induced by $V(\mathcal{Q}_n) \setminus \mathcal{F}$ with one path joining a_i and b_i, for each $i \in \{1, \dots, N+O\}$.

In [5], the smallest m defined in Problem 1.6 is denoted $[M, C, N, O]$. For example, using this notation, Havel's lemma states $[0,0,1,0] = 1$, Lemma 1.1 states $[0,0,2,0] = 2$, Lemma 1.2 states $[0,0,0,2] = 4$, Lemma 1.3 states $[0,0,3,0] = 5$, Lemma 1.4 states $[1,1,0,1] = 2$ and Lemma 1.5 states $[1,1,1,1] = 4$.

In the following table are given some of the values of $[M, C, N, O]$ known to us. Most of them were obtained in [5]. [7] contains $[0,0,1,2] = 4$ and [8] contains $[4,0,2,0] = 5$ and $[7,1,0,1] = 6$ (the latter result is not included in the table). The rows represent admissible combinations of M and C and the columns contain all the values of N and O such that $N + O \leq 3$. Each star in the table represents an impossible case. The missing entries in the table correspond to values of $[M, C, N, O]$ that we do not know yet. The inequalities in the table represent an upper or lower bound of the corresponding entry.

$MC \backslash NO$	01	10	20	11	02	30	21	12	03
00	$\star$	1	2	$\star$	4	5	$\star$	4	$\star$
11	2	$\star$	$\star$	4	$\star$	$\star$	$5^\dagger$	$\star$	$5^\dagger$
20	$\star$	4	4	$\star$	5		$\star$		$\star$
22	$\star$	$\star$	$\star$	$\star$	4	$\star$	$\star$	≤ 6	$\star$
31	4	$\star$	$\star$	5	$\star$	$\star$		$\star$	
33	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	≤ 6
40	$\star$	5	5	$\star$			$\star$		$\star$
42	$\star$	$\star$	$\star$	$\star$	5	$\star$	$\star$		$\star$
44	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
51	5	$\star$	$\star$	≥ 5	$\star$	$\star$		$\star$	

In this paper we continue to investigate the existence of path coverings with prescribed ends in hypercubes $\mathcal{Q}_n$ with one faulty vertex. More specifically, we prove the following two results: Let u be a faulty even (odd) vertex in $\mathcal{Q}_n$, u_1 and u_2 be distinct even (odd) vertices of $\mathcal{Q}_n - \{u\}$ and v_1, v_2, v_3 and v_4 be distinct odd (even) vertices of $\mathcal{Q}_n - \{u\}$. If $n \geq 5$, then there exist a $3-$path covering of $\mathcal{Q}_n - \{u\}$, with one path joining u_1 and u_2, one path joining v_1 and v_2 and one path joining v_3 and v_4. Also, under the same hypotheses, there exist a $3-$path covering of $\mathcal{Q}_n - \{u\}$, one joining u_1 and v_1, one joining u_2 and v_2 and one joining v_3 and v_4. Both claims are not always true when $n \leq 4$. In the proofs we use many of the results contained in the above table. In the notation of [5], our results are equivalent to the statements $[1, 1, 2, 1] = 5$ and $[1, 1, 0, 3] = 5$ (these values are marked with a dagger in the above table).

2. Definitions and notation

In the proofs that follow we shall use the notation already introduced in [5].

In many cases, it will be useful to view $\mathcal{Q}_n$ as two copies of Q_{n-1}, denoted $\mathcal{Q}_n^{top}$ and $\mathcal{Q}_n^{bot}$, with the vertices of $\mathcal{Q}_n^{top}$ and those of $\mathcal{Q}_n^{bot}$ joined by edges in pairs. $\mathcal{Q}_n^{top}$ and $\mathcal{Q}_n^{bot}$ will be called the *top plate* and the *bottom plate*, respectively. Without loss of generality, we may assume that the vertices of $\mathcal{Q}_n^{top}$ all share the same v-coordinate (say $v = 1$), and the vertices of $\mathcal{Q}_n^{bot}$ all share the same v-coordinate (say $v = 0$). The coordinates other than the v-coordinate will be denoted by the letters $x, y, z, \ldots$.

Any path in $\mathcal{Q}_n$ with an endpoint r can be written uniquely as (r, η), where η is a sequence of letters in $\{x, y, z, v, \ldots\}$ (such a sequence will be called a *word*). For example, in $\mathcal{Q}_5$, denote the five coordinates by x, y, z, w, v in that order. Then the path starting at $r = (0, 1, 0, 0, 1)$, going through $(1, 1, 0, 0, 1)$, $(1, 1, 0, 0, 0)$, $(1, 0, 0, 0, 0)$ and ending at $(1, 0, 1, 0, 0)$ can be written $(r, xvyz)$.

The initial vertex of the path (r, η) is r; if the terminal vertex is s, then the same path will sometimes be written $(r, \eta; s)$.

We shall also use the following notation: ω^R means the reverse word of ω; ω' denotes the word obtained after the last letter is deleted from ω; ω^* is the word obtained after the first letter is deleted from ω and $\varphi(\omega)$ is the first letter of ω.

3. $[1, 1, 0, 3] = 5$

Theorem 3.1 ($[1, 1, 0, 3] = 5$)**.** *Let $n \geq 5$, and $r, r_1, r_2, g_1, g_2, g_3, g_4$ be three distinct red and four distinct green vertices in $\mathcal{Q}_n$. Then there exists a 3−path covering $(r_1, \xi; r_2)$, $(g_1, \eta; g_2)$, $(g_3, \zeta; g_4)$ of $\mathcal{Q}_n - \{r\}$. The claim is not always true if $n < 5$.*

Proof. The following counterexample shows that $[1, 1, 0, 3] \geq 5$.

Let $n = 4$ and $r = (0, 1, 1, 0)$. Let also $r_1 = (0, 0, 1, 1)$, $r_2 = (0, 1, 0, 1)$, $g_1 = (1, 0, 1, 1)$, $g_2 = (1, 1, 1, 0)$, $g_3 = (1, 1, 0, 1)$, and $g_4 = (1, 0, 0, 0)$ be vertices in $\mathcal{Q}_4 - \{r\}$. Then one can directly verify that a 3−path covering of $\mathcal{Q}_4 - \{r\}$ with paths connecting r_1 to r_2, g_1 to g_2, and g_3 to g_4 does not exist.

Now, let $n \geq 5$. Let v be a coordinate where r_1 and r_2 do not agree, and use that coordinate to split $\mathcal{Q}_n$ into two plates. Without loss of generality, we can assume that r and r_1 are on the top plate and r_2 is on the bottom plate. There are six different cases to consider that depend on the distribution of the green terminals on the plates.

Case (1) g_1, g_2, g_3, g_4 are on the top plate.

Use $[2, 2, 0, 2] = 4$ to find a 2−path covering γ_1, γ_2 of $\mathcal{Q}_n^{top} - \{r, r_1\}$ that connects g_1 to g_2 and g_3 to g_4, respectively. Use $[0, 0, 1, 0] = 1$ to find a Hamiltonian path γ_3 of $\mathcal{Q}_n^{bot}$ that connects r_1v to r_2. Then γ_1, γ_2, $(r_1, v\gamma_3)$ is the required 3−path covering of $\mathcal{Q}_n - \{r\}$.

Case (2) g_1, g_2, g_3 are on the top plate and g_4 is on the bottom plate.

We consider two cases: $r_1v \neq g_4$ and $r_1v = g_4$.

Assume that $r_1v \neq g_4$. Choose a green vertex g in $\mathcal{Q}_n^{top} - \{g_1, g_2, g_3\}$ such that $gv \neq r_2$. Use $[2, 2, 0, 2] = 4$ to find a 2−path covering γ_1, γ_2 of $\mathcal{Q}_n^{top} - \{r, r_1\}$ that connects g_1 to g_2 and g_3 to g, respectively. Use $[0, 0, 2, 0] = 2$ to find a 2−path covering γ_3, γ_4 of $\mathcal{Q}_n^{bot}$ that connects r_1v to r_2 and gv to g_4, respectively. Then γ_1, $(g_3, \gamma_2 v\gamma_4)$, $(r_1, v\gamma_3)$ is the required 3−path covering of $\mathcal{Q}_n - \{r\}$.

Assume now that $r_1v = g_4$. Let g be a green vertex in $\mathcal{Q}_n^{top} - \{g_1, g_2, g_3\}$ adjacent to r_1. Such vertex exists since $n \geq 5$. Let $g = r_1x$ for some letter x. There are two possibilities: $gv = r_2$ and $gv \neq r_2$.

Assume that $gv = r_2$. Then $g_3v \neq r_2$. Use $[2, 2, 0, 2] = 4$ to find a 2−path covering γ_1, η of $\mathcal{Q}_n^{top} - \{r, r_1\}$ that connects g_1 to g_2 and g to g_3, respectively. Use $[0, 0, 2, 0] = 2$ to find a 2−path covering γ_2, γ_3 of $\mathcal{Q}_n^{bot}$ that connects $g\eta' v$ to r_2 and g_3v to g_4, respectively. Then γ_1, $(g_3, v\gamma_2)$, $(r_1, x\eta' v\gamma_3)$ is the required 3−path covering of $\mathcal{Q}_n - \{r\}$.

Finally, assume that $gv \neq r_2$. Use $[2, 2, 0, 2] = 4$ to find a 2−path covering γ_1, η of $\mathcal{Q}_n^{top} - \{r, r_1\}$ that connects g_1 to g_2 and g_3 to g, respectively. Use $[0, 0, 0, 2] = 4$ to find a 2−path covering γ_2, γ_3 of $\mathcal{Q}_n^{bot}$ that connects $g_3\eta' v$ to g_4 and gv to r_2, respectively. Then γ_1, $(g_3, \eta' v\gamma_2)$, $(r_1, xv\gamma_3)$ is the required 3−path covering of $\mathcal{Q}_n - \{r\}$.

Case (3) g_1, g_2 are on the top plate and g_3, g_4 are on the bottom plate.

Choose a green vertex g in $\mathcal{Q}_n^{top} - \{g_1, g_2, r_2v\}$. Use $[1,1,1,1] = 4$ to find a 2−path covering γ_1, γ_2 of $\mathcal{Q}_n^{top} - \{r\}$ that connects g_1 to g_2 and r_1 to g, respectively. Use $[0,0,0,2] = 4$ to find a 2−path covering γ_3, γ_4 of $\mathcal{Q}_n^{bot}$ that connects gv to r_2 and g_3 to g_4, respectively. Then γ_1, $(r_1, \gamma_2 v \gamma_3)$, γ_4 is the required 3−path covering of $\mathcal{Q}_n - \{r\}$.

Case (4) g_1, g_3 are on the top plate and g_2, g_4 are on the bottom plate.

Without loss of generality we can assume that $g_1v \neq r_2$ for if not then $g_3v \neq r_2$. Use $[2,0,1,0] = 4$ to find a Hamiltonian path γ of $\mathcal{Q}_n^{top} - \{r, g_1\}$ that connects r_1 to g_3. Since $n \geq 5$, the length of this path is at least 13, and so there are at least 7 edges $(\hat{r}, \hat{g})$ in γ, where $\hat{r}$ is red, $\hat{g}$ is green and $\hat{r}$ is closer to r_1 along γ than $\hat{g}$ is. Choose one such edge $(\hat{r}, \hat{g})$ with the property that $\hat{r}v \notin \{g_2, g_4\}$ and $\hat{g}v \neq r_2$. Let x be such that $\hat{g} = \hat{r}x$ and write $\gamma = \eta x \theta$, with $\hat{r} = r_1\eta$, $g_3 = \hat{g}\theta$. Use $[0,0,1,2] = 4$ to find a 3−path covering γ_1, γ_2, γ_3 of $\mathcal{Q}_n^{bot}$ that connects $\hat{r}v$ to r_2, $\hat{g}v$ to g_4, and g_1v to g_2, respectively. Then $(r_1, \eta v \gamma_1)$, $(g_3, \theta^R v \gamma_2)$, $(g_1, v\gamma_3)$ is the required 3−path covering of $\mathcal{Q}_n - \{r\}$.

Case (5) g_1 is on the top plate and g_2, g_3, g_4 are on the bottom plate.

Choose green vertices g and $\hat{g}$ in $\mathcal{Q}_n^{top} - \{g_1\}$ such that $gv \neq r_2$ and $\hat{g}v \neq r_2$. Use $[1,1,1,1] = 4$ to find a 2−path covering γ_1, γ_2 of $\mathcal{Q}_n^{top} - \{r\}$ that connects g_1 to $\hat{g}$ and r_1 to g, respectively. Use $[0,0,1,2] = 4$ to find a 3−path covering γ_3, γ_4, γ_5 of $\mathcal{Q}_n^{bot}$ that connects gv to r_2, $\hat{g}v$ to g_2, and g_3 to g_4, respectively. Then $(g_1, \gamma_1 v \gamma_4)$, $(r_1, \gamma_2 v \gamma_3)$, γ_5 is the required 3−path covering of $\mathcal{Q}_n - \{r\}$.

Case (6) g_1, g_2, g_3, g_4 are on the bottom plate.

Use $[1,1,0,1] = 2$ to find a Hamiltonian path γ of $\mathcal{Q}_n^{bot} - \{r_2\}$ that connects g_1 to g_2. We can assume that $\gamma = \eta\theta\zeta$ with $g_3 = g_1\eta$, $g_4 = g_3\theta$, by renumbering g_3 and g_4, if necessary. Use $[1,1,1,1] = 4$ to find a 2−path covering γ_1, γ_2 of $\mathcal{Q}_n^{top} - \{r\}$ that connects $g_1\eta' v$ to $g_4\varphi(\zeta)v$ and r_2v to r_1, respectively. Then θ, $(r_2, v\gamma_2)$, $(g_1, \eta' v \gamma_1 v \zeta^*)$ is the required 3−path covering of $\mathcal{Q}_n - \{r\}$. □

4. $[1,1,2,1] = 5$

Theorem 4.1 ($[1,1,2,1] = 5$). *Let $n \geq 5$ and $r, r_1, r_2, g_1, g_2, g_3, g_4$ be three distinct red and four distinct green vertices in $\mathcal{Q}_n$. Then there exists a 3−path covering $(g_1, \xi; r_1)$, $(g_2, \eta; r_2)$, $(g_3, \zeta; g_4)$ of $\mathcal{Q}_n - \{r\}$. The claim is not true if $n < 5$.*

Proof. The following counterexample shows that $[1,1,2,1] \geq 5$.

Let $n = 4$ and $r = (0,1,1,0)$. Let also $r_1 = (0,0,1,1)$, $r_2 = (0,0,0,0)$, $g_1 = (0,1,0,0)$, $g_2 = (0,1,1,1)$, $g_3 = (0,0,1,0)$, and $g_4 = (0,0,0,1)$ be vertices in $\mathcal{Q}_4 - \{r\}$. Then one can directly verify that a $3-$path covering of $\mathcal{Q}_4 - \{r\}$ with paths connecting g_1 to r_1, g_2 to r_2, and g_3 to g_4 does not exist.

Now, let $n \geq 5$. Let v be a coordinate where r_1 and r_2 do not agree, and use that coordinate to split $\mathcal{Q}_n$ into two plates. Without loss of generality, we can assume that r and r_1 are on the top plate and r_2 is on the bottom plate. There are sixteen different cases to consider that depend on the distribution of the green terminals on the plates.

Case (1) g_1, g_2, g_3, g_4 are on the top plate.

Use $[2,2,0,2] = 4$ to find a $2-$path covering $(g_1, \gamma_1; g_2)$, $(g_3, \gamma_2; g_4)$ of $\mathcal{Q}_n^{top} - \{r, r_1\}$. Either $g_1 v \neq r_2$ or $g_2 v \neq r_2$. If $g_1 v \neq r_2$ then use $[0,0,2,0] = 2$ to find a $2-$path covering $(g_1 v, \gamma_3; r_1 v)$, $(g_1 \varphi(\gamma_1) v, \gamma_4; r_2)$ of $\mathcal{Q}_n^{bot}$. Then $(g_1, v\gamma_3 v; r_1)$, $(g_2, (\gamma_1^R)' v \gamma_4; r_2)$, $(g_3, \gamma_2; g_4)$ is the required $3-$path covering of $\mathcal{Q}_n - \{r\}$. If $g_2 v \neq r_2$ then use $[0,0,0,2] = 4$ to find a $2-$path covering $(g_2 v, \gamma_3; r_2)$, $(g_1 \gamma_1' v, \gamma_4; r_1 v)$ of $\mathcal{Q}_n^{bot}$. Then $(g_1, \gamma_1' v \gamma_3 v; r_1)$, $(g_2, v\gamma_3; r_2)$, $(g_3, \gamma_2; g_4)$ is the required $3-$path covering of $\mathcal{Q}_n - \{r\}$.

Case (2) g_2, g_3, g_4 are on the top plate and g_1 is on the bottom plate.

Use $[3,1,0,1] = 4$ to find a Hamiltonian path $(g_3, \gamma_1; g_4)$ of $\mathcal{Q}_n^{top} - \{r, r_1, g_2\}$.

If $g_2 v \neq r_2$ and $r_1 v \neq g_1$ use $[0,0,0,2] = 4$ to find a $2-$path covering $(g_1, \gamma_2; r_1 v)$, $(g_2 v, \gamma_3; r_2)$ of $\mathcal{Q}_n^{bot}$. Then $(g_1, \gamma_2 v; r_1)$, $(g_2, v\gamma_3; r_2)$, $(g_3, \gamma_1; g_4)$ is the required $3-$path covering of $\mathcal{Q}_n - \{r\}$.

If $g_2 v = r_2$ and $r_1 v \neq g_1$ use $[1,1,0,1] = 2$ to find a Hamiltonian path $(g_1, \gamma_2; r_1 v)$ of $\mathcal{Q}_n^{bot} - \{r_2\}$. Then $(g_1, \gamma_2 v; r_1)$, $(g_2, v; r_2)$, $(g_3, \gamma_1; g_4)$ is the required $3-$path covering of $\mathcal{Q}_n - \{r\}$.

If $g_2 v \neq r_2$ and $r_1 v = g_1$ use $[1,1,0,1] = 2$ to find a Hamiltonian path $(g_2 v, \gamma_2; r_2)$ of $\mathcal{Q}_n^{bot} - \{g_1\}$. Then $(g_1, v; r_1)$, $(g_2, v\gamma_2; r_2)$, $(g_3, \gamma_1; g_4)$ is the required $3-$path covering of $\mathcal{Q}_n - \{r\}$.

Finally, suppose $g_2 v = r_2$ and $r_1 v = g_1$. Let x, y be the first two letters of γ, and write $\gamma_1 = xy\eta$. Then $g_3 x v \neq g_1$ and $g_3 x y v \neq r_2$, so we can use $[2,0,1,0] = 4$ to find a Hamiltonian path $(g_3 x v, \gamma_2; g_3 x y v)$ of $\mathcal{Q}_n^{bot} - \{g_1, r_2\}$. Then $(g_1, v; r_1)$, $(g_2, v; r_2)$, $(g_3, x v \gamma_2 v \eta; g_4)$ is the required $3-$path covering of $\mathcal{Q}_n - \{r\}$.

Case (3) g_1, g_3, g_4 are on the top plate and g_2 is on the bottom plate.

Use $[1,1,1,1]=4$ to find a $2-$path covering $(g_1,\gamma_1;r_1)$, $(g_3,\gamma_2;g_4)$ of $\mathcal{Q}_n^{top}-\{r\}$ and use $[0,0,1,0]=1$ to find a Hamiltonian path $(g_2,\gamma_3;r_2)$ of $\mathcal{Q}_n^{bot}$. Then $(g_1,\gamma_1;r_1)$, $(g_2,\gamma_3;r_2)$, $(g_3,\gamma_2;g_4)$ is the required $3-$path covering of $\mathcal{Q}_n-\{r\}$.

Case (4) g_1,g_2,g_4 are on the top plate and g_3 is on the bottom plate.

This case is equivalent to the next case.

Case (5) g_1,g_2,g_3 are on the top plate and g_4 is on the bottom plate.

Choose a red vertex $\hat{r}$ in $\mathcal{Q}_n^{top}-\{r,r_1,g_4v\}$. Either $g_2v\neq r_2$ or $g_3v\neq r_2$. If $g_2v\neq r_2$ then use $[2,0,2,0]=4$ to find a $2-$path covering $(g_1,\gamma_1;r_1)$, $(g_3,\gamma_2;\hat{r})$ of $\mathcal{Q}_n^{top}-\{r,g_2\}$ and use $[0,0,0,2]=4$ to find a $2-$path covering $(\hat{r}v,\gamma_3;g_4)$, $(g_2v,\gamma_4;r_2)$ of $\mathcal{Q}_n^{bot}$. Then $(g_1,\gamma_1;r_1)$, $(g_2,v\gamma_4;r_2)$, $(g_3,\gamma_2v\gamma_3;g_4)$ is the required $3-$path covering of $\mathcal{Q}_n-\{r\}$. If $g_3v\neq r_2$ then use $[2,0,2,0]=4$ to find a $2-$path covering $(g_1,\gamma_1;r_1)$, $(g_2,\gamma_2;\hat{r})$ of $\mathcal{Q}_n^{top}-\{r,g_3\}$ and use $[0,0,2,0]=2$ to find a $2-$path covering $(\hat{r}v,\gamma_3;r_2)$, $(g_3v,\gamma_4;g_4)$ of $\mathcal{Q}_n^{bot}$. Then $(g_1,\gamma_1;r_1)$, $(g_2,\gamma_2v\gamma_3;r_2)$, $(g_3,v\gamma_4;g_4)$ is the required $3-$path covering of $\mathcal{Q}_n-\{r\}$.

Case (6) g_1,g_2 are on the top plate and g_3,g_4 are on the bottom plate.

Choose a green vertex g in $\mathcal{Q}_n^{top}-\{g_1,g_2\}$ such that $gv\neq r_2$. Use $[1,1,1,1]=4$ to find a $2-$path covering $(g_1,\gamma_1;r_1)$, $(g_2,\gamma_2;g)$ of $\mathcal{Q}_n^{top}-\{r\}$. Use $[0,0,0,2]=4$ to find a $2-$path covering $(gv,\gamma_3;r_2)$, $(g_3,\gamma_4;g_4)$ of $\mathcal{Q}_n^{bot}$. Then $(g_1,\gamma_1;r_1)$, $(g_2,\gamma_2v\gamma_3;r_2)$, $(g_3,\gamma_4;g_4)$ is the required $3-$path covering of $\mathcal{Q}_n-\{r\}$.

Case (7) g_1,g_3 are on the top plate and g_2,g_4 are on the bottom plate.

This case is equivalent to the next case.

Case (8) g_1,g_4 are on the top plate and g_2,g_3 are on the bottom plate.

Choose a green vertex g in $\mathcal{Q}_n^{top}-\{g_1,g_4\}$ such that $gv\neq r_2$. Use $[1,1,1,1]=4$ to find a $2-$path covering $(g_1,\gamma_1;r_1)$, $(g,\gamma_2;g_4)$ of $\mathcal{Q}_n^{top}-\{r\}$. Use $[0,0,2,0]=2$ to find a $2-$path covering $(g_3,\gamma_3;gv)$, $(g_2,\gamma_4;r_2)$ of $\mathcal{Q}_n^{bot}$. Then $(g_1,\gamma_1;r_1)$, $(g_2,\gamma_4;r_2)$, $(g_3,\gamma_3v\gamma_2;g_4)$ is the required $3-$path covering of $\mathcal{Q}_n-\{r\}$.

Case (9) g_2,g_3 are on the top plate and g_1,g_4 are on the bottom plate.

This case is equivalent to the next case.

Case (10) g_2,g_4 are on the top plate and g_1,g_3 are on the bottom plate.

Either $g_2v\neq r_2$ or $g_4v\neq r_2$.

If $g_2v\neq r_2$ then use $[2,0,1,0]=4$ to find a Hamiltonian path $(r_1,\gamma;g_4)$ of $\mathcal{Q}_n^{top}-\{r,g_2\}$. Since $n\geq 5$, the length of this path is at least 13, and so there are at least 7 edges $(\hat{r},\hat{g})$ in γ, where $\hat{r}$ is red, $\hat{g}$ is green and $\hat{r}$ is closer to r_1 along γ than $\hat{g}$ is. Choose one such edge $(\hat{r},\hat{g})$ with the property that $\hat{r}v \notin \{g_1, g_3\}$ and $\hat{g}v \neq r_2$. Let x be such that $\hat{g} = \hat{r}x$ and write $\gamma = \eta x \theta$, with $\hat{r} = r_1\eta$, $g_4 = \hat{g}\theta$. Use $[0,0,1,2] = 4$ to find a 3−path covering $(g_1, \gamma_1; \hat{r}v)$, $(g_2v, \gamma_2; r_2)$, $(g_3, \gamma_3; \hat{g}v)$ of $\mathcal{Q}_n^{bot}$. Then $(g_1, \gamma_1 v\eta^R; r_1)$, $(g_2, v\gamma_2; r_2)$, $(g_3, \gamma_3 v\theta; g_4)$ is the required 3−path covering of $\mathcal{Q}_n - \{r\}$.

If $g_4v \neq r_2$ then use $[2,0,1,0] = 4$ to find a Hamiltonian path $(r_1, \gamma; g_2)$ of $\mathcal{Q}_n^{top} - \{r, g_4\}$. Since $n \geq 5$, the length of this path is at least 13, and so there are at least 7 edges $(\hat{r},\hat{g})$ in γ, where $\hat{r}$ is red, $\hat{g}$ is green and $\hat{r}$ is closer to r_1 along γ than $\hat{g}$ is. Choose one such edge $(\hat{r},\hat{g})$ with the property that $\hat{r}v \notin \{g_1, g_3\}$ and $\hat{g}v \neq r_2$. Let x be such that $\hat{g} = \hat{r}x$ and write $\gamma = \eta x\theta$, with $\hat{r} = r_1\eta$, $g_2 = \hat{g}\theta$. Use $[0,0,1,2] = 4$ to find a 3−path covering $(g_1, \gamma_1; \hat{r}v)$, $(\hat{g}v, \gamma_2; r_2)$, $(g_3, \gamma_3; g_4v)$ of $\mathcal{Q}_n^{bot}$. Then $(g_1, \gamma_1 v\eta^R; r_1)$, $(g_2, \theta^R v\gamma_2; r_2)$, $(g_3, \gamma_3 v; g_4)$ is the required 3−path covering of $\mathcal{Q}_n - \{r\}$.

Case (11) g_3, g_4 are on the top plate and g_1, g_2 are on the bottom plate.

Choose a green vertex g in $\mathcal{Q}_n^{top} - \{g_3, g_4\}$ such that $gv \neq r_2$. Use $[1,1,1,1] = 4$ to find a 2−path covering $(g, \gamma_1; r_1)$, $(g_3, \gamma_2; g_4)$ of $\mathcal{Q}_n^{top} - \{r\}$. Use $[0,0,2,0] = 2$ to find a 2−path covering $(g_1, \gamma_3; gv)$, $(g_2, \gamma_4; r_2)$ of $\mathcal{Q}_n^{bot}$. Then $(g_1, \gamma_3 v\gamma_1; r_1)$, $(g_2, \gamma_4; r_2)$, $(g_3, \gamma_2; g_4)$ is the required 3−path covering of $\mathcal{Q}_n - \{r\}$.

Case (12) g_1 is on the top plate and g_2, g_3, g_4 are on the bottom plate.

Choose two distinct green vertices g and $\hat{g}$ in $\mathcal{Q}_n^{top} - \{g_1\}$ such that $gv \neq r_2$ and $\hat{g}v \neq r_2$. Use $[1,1,1,1] = 4$ to find a 2−path covering $(g_1, \gamma_1; r_1)$, $(g, \gamma_2; \hat{g})$ of $\mathcal{Q}_n^{top} - \{r\}$. Use $[0,0,1,2] = 4$ to find a 3−path covering (g_2, γ_3, gv), $(\hat{g}v, \gamma_4; r_2)$, $(g_3, \gamma_5; g_4)$ of $\mathcal{Q}_n^{bot}$. Then $(g_1, \gamma_1; r_1)$, $(g_2, \gamma_3 v\gamma_2 v\gamma_4; r_2)$, $(g_3, \gamma_5; g_4)$ is the required 3−path covering of $\mathcal{Q}_n - \{r\}$.

Case (13) g_2 is on the top plate and g_1, g_3, g_4 are on the bottom plate.

Choose two distinct green vertices g and $\hat{g}$ in $\mathcal{Q}_n^{top} - \{g_1\}$ such that $gv \neq r_2$ and $\hat{g}v \neq r_2$. Use $[1,1,1,1] = 4$ to find a 2−path covering $(g, \gamma_1; r_1)$, $(g_2, \gamma_2; \hat{g})$ of $\mathcal{Q}_n^{top} - \{r\}$. Use $[0,0,1,2] = 4$ to find a 3−path covering (g_1, γ_3, gv), $(\hat{g}v, \gamma_4; r_2)$, $(g_3, \gamma_5; g_4)$ of $\mathcal{Q}_n^{bot}$. Then $(g_1, \gamma_3 v\gamma_1; r_1)$, $(g_2, \gamma_2 v\gamma_4; r_2)$, $(g_3, \gamma_5; g_4)$ is the required 3−path covering of $\mathcal{Q}_n - \{r\}$.

Case (14) g_3 is on the top plate and g_1, g_2, g_4 are on the bottom plate.

This case is equivalent to the next case.

Case (15) g_4 is on the top plate and g_1, g_2, g_3 are on the bottom plate.

Let $x \neq v$ be a letter such that $g_3xv \neq g_4$ and $\hat{r} \neq r_2$ be a red vertex in $\mathcal{Q}_n^{bot}$ such that $\hat{r}v \neq g_4$. Use $[1,1,1,1] = 4$ to find a $2-$ path covering $(\hat{r}v, \gamma_1; r_1)$, $(g_3xv, \gamma_2; g_4)$ of $\mathcal{Q}_n^{top} - \{r\}$. Use $[2,0,2,0] = 4$ to find a $2-$path covering $(g_1, \gamma_3, \hat{r})$, $(g_2, \gamma_4; r_2)$ of $\mathcal{Q}_n^{bot} - \{g_3, g_3x\}$. Then $(g_1, \gamma_3 v \gamma_1; r_1)$, $(g_2, \gamma_4; r_2)$, $(g_3, xv\gamma_2; g_4)$ is the required $3-$path covering of $\mathcal{Q}_n - \{r\}$.

Case (16) g_1, g_2, g_3, g_4 are on the bottom plate.

Use $[1,1,0,1] = 2$ to find a Hamiltonian path $(g_1, \gamma; g_2)$ of $\mathcal{Q}_n^{bot} - \{r_2\}$. By renumbering g_3 and g_4, if necessary, we can assume that $\gamma = \eta\theta\zeta$ with $g_3 = g_1\eta$, $g_4 = g_3\theta$. Use $[1,1,1,1] = 4$ to find a $2-$path covering $(g_1\eta' v, \gamma_1; r_1)$, $(g_2(\zeta^R)'v, \gamma_2; r_2 v)$ of $\mathcal{Q}_n^{top} - \{r\}$. Then $(g_1, \eta' v \gamma_1; r_1)$, $(g_2, (\zeta^R)' v \gamma_2 v; r_2)$, $(g_3, \theta; g_4)$ is the required $3-$path covering of $\mathcal{Q}_n - \{r\}$. □

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