

On the continuity of factorizations

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ABSTRACT. Let $\{X_i : i \in I\}$ be a set of sets, $X_J := \prod_{i \in J} X_i$ when $\emptyset \neq J \subseteq I$; Y be a subset of X_I , Z be a set, and $f : Y \rightarrow Z$. Then f is said to *depend on J* if $p, q \in Y$, $p_J = q_J \Rightarrow f(p) = f(q)$; in this case, $f_J : \pi_J[Y] \rightarrow Z$ is well-defined by the rule $f = f_J \circ \pi_J|_Y$.

When the X_i and Z are spaces and $f : Y \rightarrow Z$ is continuous with Y dense in X_I , several natural questions arise:

- (a) does f depend on some small $J \subseteq I$?
- (b) if it does, when is f_J continuous?
- (c) if f_J is continuous, when does it extend to continuous $\overline{f}_J : X_J \rightarrow Z$?
- (d) if f_J so extends, when does f extend to continuous $\overline{f} : X_I \rightarrow Z$?
- (e) if f depends on some $J \subseteq I$ and f extends to continuous $\overline{f} : X_I \rightarrow Z$, when does \overline{f} also depend on J ?

The authors offer answers (some complete, some partial) to some of these questions, together with relevant counterexamples.

Theorem 1. f has a continuous extension $\overline{f} : X_I \rightarrow Z$ that depends on J if and only if f_J is continuous and has a continuous extension $\overline{f}_J : X_J \rightarrow Z$.

Example 1. For $\omega \leq \kappa \leq \mathfrak{c}$ there are a dense subset Y of $[0, 1]^\kappa$ and $f \in C(Y, [0, 1])$ such that f depends on every nonempty $J \subseteq \kappa$, there is no $J \in [\kappa]^{<\omega}$ such that f_J is continuous, and f extends continuously over $[0, 1]^\kappa$.

Example 2. There are a Tychonoff space X_I , dense $Y \subseteq X_I$, $f \in C(Y)$, and $J \in [I]^{<\omega}$ such that f depends on J , $\pi_J[Y]$ is C -embedded in X_J , and f does not extend continuously over X_I .

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1. INTRODUCTION

In addition to the notation given in the Abstract, we adopt these conventions. ω is (the cardinality of) the set of non-negative integers and I is an index set (usually infinite). α, κ , and λ are cardinals and $[I]^{<\kappa} := \{J \subseteq I : |J| < \kappa\}$. $(X_I)_\kappa$ denotes $X_I := \prod_{i \in I} X_i$ with the κ -box topology (so $(X_I)_\omega = X_I$) and $\Sigma_\lambda(p) := \{x \in X_I : |\{i \in I : x_i \neq p_i\}| < \lambda\}$ whenever $p \in X_I$. By a (canonical) basic open set in $(X_I)_\kappa$ we mean a set of the form $U = U_I = \prod_{i \in I} U_i$ with U_i

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open in X_i and with $R(U) := \{i \in I : U_i \neq X_i\} \in [I]^{<\kappa}$. (In the terminology of [4], $R(U)$ is the *restriction set* of the (basic) open set U .) The symbol \mathbb{R} denotes the real line with its usual topology, the cardinality of \mathbb{R} is denoted by \mathfrak{c} , the cardinality of the set X is denoted by $|X|$, and the closure of X by \bar{X} .

The *weight* of a space X is $w(X) := \min\{|\mathcal{B}| : \mathcal{B} \text{ a base for } X\} + \omega$, the *density* of X is $d(X) := \min\{|D| : D \text{ is dense in } X\} + \omega$, $\chi(x, X)$ denotes the *character* (i.e., the local weight) of the point x in the space X , and $\chi(X) := \sup\{\chi(x, X) : x \in X\}$. Finally, a pairwise disjoint collection of nonempty open sets in X is called a *cellular family* and the *cellularity* of X is $c(X) := \sup\{|\mathcal{U}| : \mathcal{U} \text{ a cellular family in } X\} + \omega$.

For spaces Y and Z we denote by $C(Y, Z)$ the set of continuous functions from Y into Z . We write $C(X) := C(X, \mathbb{R})$ and (in contrast with the convention used in [8] and elsewhere) we write $C^*(X) := C(X, [0, 1])$. A subspace Y of a space X is *$C(Z)$ -embedded* if every $f \in C(Y, Z)$ extends to $\bar{f} \in C(X, Z)$; then as usual [8], a *$C(Z)$ -embedded space* $Y \subseteq X$ with $Z = \mathbb{R}$ is said to be *C -embedded*; if $Z = [0, 1]$ then Y is said to be *C^* -embedded*.

Our spaces are not subjected to any standing separation hypothesis. When a specific property is wanted, as in Remark 3.4, Section 4, and Theorems 5.5 and 5.6, we state it explicitly.

Definition 1.1. *When X_I, Y, Z and f are as in the Abstract and f depends on $J \subseteq I$, the function $f_J : \pi_J[Y] \rightarrow Z$ (defined by the relation $f = f_J \circ \pi_J|_Y$) is a factorization of f .*

For additional topological definitions not given above, see [8], [4], [16], [11], or [7].

The point of departure of our investigation is a lemma given in the book “Chain Conditions in Topology” by W. W. Comfort and S. Negreponitis [4], together with a question those authors posed. We give these now, paraphrasing slightly to facilitate the present exposition.

Lemma 1.2. [4, 10.3] *Let $\omega \leq \kappa \leq \alpha$, $\{X_i : i \in I\}$ be a set of nonempty topological spaces, and Y be a subspace of $(X_I)_\kappa$ such that $\pi_J[Y] = X_J$ for every nonempty $J \subset I$ with $|J| < \alpha$. Let also Z be a topological space and f be a continuous function from Y to Z such that f depends on $< \alpha$ coordinates. Then there is continuous $\bar{f} : (X_I)_\kappa \rightarrow Z$ such that $f \subset \bar{f}$.*

Question 1.3. [4, p. 235] *Let $\omega \leq \kappa \leq \alpha$, $\{X_i : i \in I\}$ be a set of nonempty topological spaces, and Y be a dense subspace of $(X_I)_\kappa$. Let Z be a space such that $\pi_J[Y]$ is $C(Z)$ -embedded in X_J for every nonempty $J \in [I]^{<\alpha}$. If $f \in C(Y, Z)$ and there is $J \in [I]^{<\alpha}$ such that f depends on J , must f extend continuously over $(X_I)_\kappa$? What about the case $\kappa = \omega$?*

In fact, the present authors do not know the answer to Question 1.3 even if all the spaces X_i are assumed metrizable.

Let us specialize to the case $\kappa = \omega$. It is clear from Theorem 3.5(c) below that if the function f in Lemma 1.2 depends on some $J \in [I]^{<\alpha}$ then the factorization $f_J : X_J \rightarrow Z$ is continuous. Then since $\pi_J[Y] = X_J$, the function

$\bar{f} := f_J \circ \pi_J$ is a continuous extension of f that depends on J . Thus Lemma 1.2 has this consequence.

Theorem 1.4. *Let $\alpha \geq \omega$, X_I be a product space, and Y be a subspace of X_I such that $\pi_J[Y] = X_J$ for every nonempty $J \in [I]^{<\alpha}$. Let also Z be a topological space and $f \in C(Y, Z)$ depend on some $J \in [I]^{<\alpha}$. Then f_J is continuous and $\bar{f} := f_J \circ \pi_J : X_I \rightarrow Z$ is a continuous extension of f that depends on J .*

Questions (a) through (e) of the Abstract are subsidiary to a more compelling very general question: “When is a continuous function defined on a dense subset of a product space continuously extendable over the full product?” This question and question (a) of our Abstract have generated a huge literature. Among the works in this vein, representing a variety of approaches, we mention these familiar papers: H. Corson [5], R. Engelking [6], I. Glicksberg [9], M. Hušek [12], [13], [14], A. Mishchenko [18], N. Noble [19], N. Noble and M. Ulmer [20], M. Ulmer [22], [23]. The textbooks [7] and [4] strive for comprehensive bibliographies. As is indicated in [4], many of the published results responding positively to these questions generalize to product spaces with the κ -box topology.

In contrast, questions (b) through (e) of the Abstract have been almost totally ignored in the literature. In this paper, always with $f \in C(Y, Z)$ and usually with Y dense in X_I , we study some of these questions and their relation to Question 1.3. In Section 2 we give some examples of discontinuous factorizations f_J ; in Section 3 we give some sufficient conditions for the existence of continuous factorizations; in Section 4 we study when the existence of a factorization of a function defined on a dense subspace of a product space implies the existence of a factorization of its continuous extension to the full product; in Section 5 we give some sufficient conditions for a positive answer to Question 1.3; and in Section 6 we pose a question related to Question 1.3.

2. DISCONTINUOUS FACTORIZATIONS: SOME EXAMPLES

In this section, responding to question (b) of the Abstract, we give examples showing that a factorization of a continuous function defined on a dense subset of a product space need not be continuous (2.3, 2.8, 2.9); and when it is continuous the initial function may (2.8) or may not (2.13) extend continuously over the full product. Concerning question (e) of the Abstract, 2.3 and 2.9 give examples of a function $f \in C(Y, Z)$ with an extension $\bar{f} \in C(X_I, Z)$ such that, for certain $J \subseteq I$, f does and \bar{f} does not depend on J . As to question (d), the answer is “Always” (4.6).

We use in what follows the familiar fact (see for example [7, 2.3.15] or [3, 3.18]) that for $\kappa \leq \mathfrak{c}$ the product of κ -many separable spaces is separable. For convenience we give statements with each $X_i = [0, 1]$, but routine generalizations (for example, with each $X_i = \mathbb{R}$) are clearly valid.

Lemma 2.1. *Let I be an index set with $0 < |I| \leq \mathfrak{c}$ and $X_I = [0, 1]^I$. Let $D = \{x(n) : n < \omega\}$ be a (countable) dense subset of X_I , and for $n < \omega$ let*

$y(n) \in X_I$ satisfy $|y(n)_i - x(n)_i| < \frac{1}{n+1}$ for each $i \in I$. Then $E := \{y(n) : n < \omega\}$ is dense in X_I .

Proof. Each nonempty open $U \subseteq X_I$ contains a (basic) set of the form

$$N(p, F, \epsilon) := \prod_{i \in F} (p_i - \epsilon, p_i + \epsilon) \times [0, 1]^{I \setminus F}$$

with $p \in X_I$, $F \in [I]^{<\omega}$ and $\epsilon > 0$. The open set $V := N(p, F, \frac{\epsilon}{2})$ satisfies $|V \cap D| = \omega$, and with n chosen so that $\frac{1}{n+1} < \frac{\epsilon}{2}$ and $x(n) \in V$ we have $y(n) \in N(p, F, \epsilon) \cap E \subseteq U \cap E$. \square

Lemma 2.2. *Let I be an index set with $0 < |I| \leq \mathfrak{c}$ and $X_I = [0, 1]^I$. There is a countable dense subset E of X_I such that for each $i \in I$ the restriction $\pi_i|E : E \rightarrow [0, 1]_i = [0, 1]$ is an injection.*

Proof. Let $D = \{x(n) : n < \omega\}$ be dense in X_I . Define $y(0)_i = x(0)_i$ for each i , and if $y(n)_i$ has been defined for all $n < m$ choose $y(m)_i \in [0, 1]$ so that $y(m)_i \notin \{y(0)_i, \dots, y(m-1)_i\}$ and $|y(m)_i - x(m)_i| < \frac{1}{m+1}$. Then by Lemma 2.1 the set $E := \{y(n) : n < \omega\}$ is as required. \square

Example 2.3. There is a product space X_I , a (countable) dense subspace $Y = \{y(n) : n < \omega\} \subseteq X_I$, $f \in C(Y)$, and an index $j \in I$, such that

- (i) the function $f_j : \pi_j[Y] \rightarrow \mathbb{R}$ given by $f_j(y(n)_j) := f(y(n))$ is well-defined, and
- (ii) f_j is not continuous on $\pi_j[Y]$.

One may arrange in addition that $f : Y \rightarrow \mathbb{R}$ extends continuously over X_I .

Proof. Let each $X_i = \mathbb{R}$ or $X_i = [0, 1]$ with $|I| = \mathfrak{c}$, and set $X_I := \prod_{i \in I} X_i$ and $Y := E = \{y(n) : n < \omega\}$ as in Lemma 2.2. Choose and fix two different coordinates $i, j \in I$ and define $f : Y \rightarrow \mathbb{R}$ by $f := \pi_i|_Y$. Note that f extends continuously over X_I . Note also that the function $f_j : \pi_j[Y] \rightarrow \mathbb{R}$ given by $f_j(y(n)_j) := f(y(n)) = y(n)_i$ is well-defined, since if $y(n), y(m) \in Y$ with $y(n)_j = y(m)_j$ then $m = n$. Now choose distinct $x, y \in Y$ and let $p = x_i$ and $q = y_j$. Thus $f_j(q) \neq p$. Let also n_k be a sequence such that $y(n_k)_i \rightarrow p$, $y(n_k)_j \rightarrow q$. Then $f_j(y(n_k)_j) = f(y(n_k)) = y(n_k)_i \rightarrow p \neq f_j(q)$, so f_j is not continuous. \square

Theorem 2.4. *Let I be an index set with $0 < |I| \leq \mathfrak{c}$ and $X_I = [0, 1]^I$. There is a dense subspace Y of X_I such that for each $i \in I$ the restriction $\pi_i|Y : Y \rightarrow [0, 1]_i = [0, 1]$ is a bijection onto $[0, 1]$.*

Proof. Begin with $E = \{y(n) : n < \omega\}$ as in Lemma 2.2 and for $i \in I$ let $\{y(\eta)_i : \omega \leq \eta < \mathfrak{c}\}$ be a faithful enumeration of the set $[0, 1] \setminus \pi_i[E]$. Then $y(\eta) \in X_I$ and the set $Y := \{y(\eta) : \eta < \mathfrak{c}\} = E \cup \{y(\eta) : \omega \leq \eta < \mathfrak{c}\}$ is as required. \square

Remark 2.5. In Example 2.3 the set $\pi_j[Y]$ is a countable, dense subspace of $\mathbb{R}_j = \mathbb{R}$ or of $[0, 1]_j = [0, 1]$, hence is not C^* -embedded. If we take I, X_I , and Y as in Theorem 2.4 with $I = \{i, j\}$, $|Y| = \mathfrak{c}$, $X_I = \mathbb{R}^2$ and with both

π_j and π_i surjections from Y onto $X_j = \mathbb{R}$ and $X_i = \mathbb{R}$, respectively, then we can arrange the essential features of that argument. Note that the function $f_j : \pi_j[Y] \rightarrow \mathbb{R}$ given by $f_j(y(\eta)_j) := f(y(\eta)) = y(\eta)_i$ is still well-defined and discontinuous since its restriction to the countable subspace E is discontinuous, though $f_j \circ \pi_j = f$ with $f(y(\eta)) = y(\eta)_i$ is continuous on Y and as before, f extends continuously over all of \mathbb{R}^2 .

In Example 2.3 the function f depends on the set $\{i\}$ and also depends on the set $\{j\}$ but the function f_i is continuous while the function f_j is not. The following proposition shows, more generally, that if (continuous) $f : Y \rightarrow Z$ depends on nonempty disjoint sets $J_1, J_2 \subset J$ with f_{J_1} continuous, then either f is a constant function or f_{J_2} is nowhere continuous.

Proposition 2.6. *Let X_I be a product space, Z be a Hausdorff space, and Y be a dense subspace of X_I . Let also J_1 and J_2 be nonempty disjoint subsets of I , $f : Y \rightarrow Z$ be a non-constant (continuous) function that depends on J_1 and J_2 , and f_{J_1} is continuous. Then f_{J_2} is discontinuous at every point of $\pi_{J_2}[Y]$.*

Proof. Let $y \in Y$. We shall show that f_{J_2} is discontinuous at y_{J_2} . Let $f_{J_2}(y_{J_2}) = z_1$, hence $f(y) = z_1$ and $f_{J_1}(y_{J_1}) = z_1$. Since the function f is not constant there is $x \in Y$ such that $f(x) \neq z_1$. Let $f(x) = z_2$, hence $f_{J_1}(x_{J_1}) = z_2$. Since Z is a Hausdorff space we can find two disjoint open sets U_1 and U_2 in Z such that $z_1 \in U_1$ and $z_2 \in U_2$. The function f_{J_1} is continuous at x_{J_1} . Therefore there is a basic open neighborhood V_{J_1} of x_{J_1} in X_{J_1} such that $f_{J_1}[V_{J_1} \cap \pi_{J_1}[Y]] \subset U_2$. Now, assume that there exists a basic open neighborhood V_{J_2} of y_{J_2} in X_{J_2} such that $f_{J_2}[V_{J_2} \cap \pi_{J_2}[Y]] \subset U_1$. Since Y is dense in X_I and $J_1 \cap J_2 = \emptyset$ there is $t \in Y$ such that $t \in \pi_{J_1}^{-1}[V_{J_1}] \cap \pi_{J_2}^{-1}[V_{J_2}]$, hence $t_{J_1} \in V_{J_1} \cap \pi_{J_1}[Y]$ and $t_{J_2} \in V_{J_2} \cap \pi_{J_2}[Y]$. Then $f(t) = f_{J_1}(t_{J_1}) = f_{J_2}(t_{J_2})$, so $f(t) \in U_1 \cap U_2 = \emptyset$, a contradiction. \square

Corollary 2.7. *Let X_I be a product space, Z be a Hausdorff space, and Y be a dense subspace of X_I . Let also J_1 and J_2 be nonempty disjoint subsets of I , $f : Y \rightarrow Z$ be a (continuous) function that depends on J_1 and J_2 , and f_{J_1} and f_{J_2} are continuous. Then f is constant.*

Example 2.3 suggests the speculation that if some $f \in C(Y, Z)$ depends on a set J with f_J discontinuous, and if f extends continuously over X_I , then one can find another set $J_1 \subset I$ with $|J_1| = |J|$ such that f depends on J_1 and f_{J_1} is continuous. We show in Corollary 2.9 below that this can fail: there exist a cardinal number $\alpha \geq \omega$, a product space X_I , a Hausdorff space Z , a dense subspace Y of X_I , and a continuous function $f : Y \rightarrow Z$ such that for every $J \in [I]^{<\alpha}$, f depends on J but f_J is not continuous, and the function f can be extended to a continuous function \bar{f} on X_I . (There is, however, a fragment of Question 1.3—logically, an equivalent formulation—which survives in the face of Corollary 2.9. We give the statement in 6.1 below.)

In what follows we take I , X_I , and Y as in Theorem 2.4 with $|I| \geq \omega$, $|Y| = \mathfrak{c}$, and with each $\pi_i|_Y : Y \rightarrow [0, 1]$ a bijection onto $[0, 1]$; the indexing $Y = \{y(\eta) : \eta < \mathfrak{c}\}$ plays no further role. We note that for every set Z , every

function $f : Y \rightarrow Z$ depends (vacuously) on each set $\{i\}$ with $i \in I$. Thus, for every $f : Y \rightarrow Z$ and $\emptyset \neq J \subseteq I$ the function $f_J : \pi_J[Y] \rightarrow Z$ is well-defined by the rule $f_J(p_J) := f(p)$ ($p \in Y$).

Theorem 2.8. *Given $Y \subseteq X_I = [0, 1]^I$ as above, let $C = \{i_n : n < \omega\} \in [I]^\omega$ (faithfully indexed) and define $g : X_I \rightarrow [0, 1]$ by $g(x) := \sum_{n < \omega} \frac{x_{i_n}}{2^n} = \sum_{n < \omega} \frac{\pi_{i_n}(x)}{2^n}$. Then for $\emptyset \neq J \subset I$ the function $g_J : \pi_J[Y] \rightarrow [0, 1]$ is continuous if and only if $C \subseteq J$.*

Proof. (We note that the series defining g converges uniformly on X_I , so $g : X_I \rightarrow [0, 1]$ is continuous.)

If. Let $p_J \in \pi_J[Y] \subseteq X_J$ with $p \in Y$, and let $p(\lambda)_J$ be a net in $\pi_J[Y]$ (with $p(\lambda) \in Y$) such that $p(\lambda)_J \rightarrow p_J$. Then $p(\lambda)_i \rightarrow p_i$ for each $i = i_n \in C$, so $g(p(\lambda)) \rightarrow g(p)$, i.e., $g_J(p(\lambda)_J) \rightarrow g_J(p_J)$.

Only if. We show (when $C \subseteq J$ fails) that g_J is continuous at no point $p_J \in \pi_J[Y]$.

Fix $\bar{i} = i_{\bar{n}} \in C \setminus J$ and $p \in Y$ and define $q \in X_I$ by $q_i := p_i$ if $\bar{i} \neq i \in I$, $|q_{i_{\bar{n}}} - p_{i_{\bar{n}}}| = \frac{1}{4}$. Since $q_i = p_i$ for all $i \neq i_{\bar{n}}$, in particular for all $i \in C \setminus \{i_{\bar{n}}\}$, we have $|g(p) - g(q)| = \frac{1}{4} \cdot \frac{1}{2^{\bar{n}}}$. There is a net $p(\lambda)$ in Y such that $p(\lambda) \rightarrow q$. Since $p_J = q_J$ we have $p(\lambda)_J \rightarrow q_J = p_J \in \pi_J[Y]$, but from the continuity of g on X_I we have $g_J(p(\lambda)_J) = g(p(\lambda)) \rightarrow g(q) \neq g(p) = g_J(p_J)$. Thus f_J is not continuous on $\pi_J[Y]$. \square

Corollary 2.9. *For $\omega \leq \kappa \leq \mathfrak{c}$ there are a dense subset Y of $[0, 1]^\kappa$ and continuous $f : Y \rightarrow [0, 1]$ such that*

- (a) f depends on every nonempty $J \subseteq \kappa$;
- (b) each restricted projection $\pi_\eta|_Y : Y \rightarrow [0, 1]_\eta = [0, 1]$ ($\eta < \kappa$) is a bijection onto $[0, 1]$;
- (c) there is no $J \in [\kappa]^{<\omega}$ such that $f_J : \pi_J[Y] \rightarrow [0, 1]$ is continuous; and
- (d) f extends to a continuous function $\bar{f} : [0, 1]^\kappa \rightarrow [0, 1]$.

Proof. From Theorem 2.8, taking $f := g|_Y$. \square

The examples just given show that a function $g \in C(X_I, Z)$ may fail to depend on a set J , even when $f := g|_Y : Y \rightarrow Z$ does depend on J . We discuss this in greater detail in Section 4.

Discussion 2.10. We draw the reader's attention to two hypotheses in Question 1.3.

- (a) there is $J \in [I]^{<\alpha}$ such that f depends on J ; and
- (b) $\pi_J[Y]$ is $C(Z)$ -embedded in X_J for every nonempty $J \in [I]^{<\alpha}$.

We end this section with examples showing that if either (a) or (b) is not satisfied then the resulting weaker questions can be answered in the negative. Again we specify to the case $\kappa = \omega$.

Example 2.11. There are $\alpha \geq \omega$, a Tychonoff product space X_I , a dense subspace $Y \subset X_I$, and $f \in C(Y)$ such that $\pi_J[Y]$ is C -embedded in X_J for all nonempty $J \in [I]^{<\alpha}$, but f has no continuous extension from X_I to \mathbb{R} and f does not depend on any proper nonempty subset $J \in [I]^{<\alpha}$.

Proof. We are aware of two relevant constructions from the literature.

(1) [the case $\alpha = \omega$] Let \mathbb{N} denote the countably infinite discrete space. It is shown in [1] that there is a sequence $\{Y_k : k < \omega\}$ of spaces, with in each case $\mathbb{N} \subseteq Y_k \subseteq \beta(\mathbb{N}) := X_k$, such that $Y := \prod_{k < \omega} Y_k$ is not pseudocompact but $Y_J = \prod_{k \in J} Y_k$ is pseudocompact whenever $J \in [\omega]^{<\omega}$. It follows from Glicksberg's theorem [9] that $\beta(Y_J) = \prod_{k \in J} X_k$ for each $J \in [\omega]^{<\omega}$ (so $\pi_J[Y] = Y_J$ is C^* -embedded in X_J), but the relation $\beta(Y) = \prod_{k < \omega} X_k$ fails (so some $f \in C^*(Y)$ has no continuous extension from $X_I = \prod_{k < \omega} X_k$ to \mathbb{R}).

If f depends on some nonempty $J \in [\omega]^{<\omega}$ then (since Y is a product space) Theorem 3.5(a) shows $f_J \in C^*(Y_J)$. Then f_J extends to $\overline{f_J} \in C^*(X_J)$ and we have $f \subseteq \overline{f_J} \circ \pi_J \in C^*(X_I)$, a contradiction.

(2) [arbitrary $\alpha \geq \omega$] Ulmer [22], [23] has given many examples, enhanced and extended in our work [2], of a Tychonoff product space X_I and a Σ_α -product subspace $Y \subset X_I$ which is not C -embedded in X_I (so some $f \in C(Y)$ has no continuous extension from X_I to \mathbb{R}).

If f depends on some nonempty $J \in [I]^{<\alpha}$ then $f_J \in C(\pi_J[Y])$ since Y is a Σ_α -space (see Theorem 3.5(b)). $\pi_J[Y]$ is trivially C -embedded in X_J since $\pi_J[Y] = X_J$, and we have the contradiction $f \subseteq \overline{f_J} \circ \pi_J \in C(X_I)$. \square

Example 2.12. There are a Tychonoff product space $X_I = \prod_{i \in I} X_i$, a dense subspace $Y \subset X_I$, a function $f \in C(Y)$, and $J \in [I]^{<\omega}$ such that

- (a) f depends on J ,
- (b) $\pi_J[Y]$ is C -embedded in X_J , and
- (c) f does not extend continuously over X_I .

Proof. In [2, 3.2] we have shown, extending arguments introduced by Ulmer [22], [23], that there are a Tychonoff space $X_{I'} = \prod_{i \in I'} X_i$ with $|X_i| = |I'| = \omega$ for each $i \in I'$, $q' \in X_{I'}$, and a continuous function $f' : X_{I'} \setminus \{q'\} \rightarrow [0, 1]$ which does not extend continuously over $X_{I'}$. We arrange the notation so that no symbol in I' is named 0, and we set $I := I' \cup \{0\}$ and $X_0 := [0, 1]$. Since $\{q'\} \times X_0$ is closed and nowhere dense in X_I , there is a countable dense subset $\{x(n) : n < \omega\}$ of X_I which misses $\{q'\} \times X_0$, and a routine modification of the argument in Lemmas 2.1 and 2.2 gives a dense set $E = \{y(n) : n < \omega\}$, also missing $\{q'\} \times X_0$, such that the restricted projection $\pi_0|_E : E \rightarrow X_0$ is an injection. (Arguing recursively one lets $y(m)_{I'} = x(m)_{I'}$ and one chooses (distinct) points $y(m)_0 \in [0, 1]$ such that $|y(m)_0 - x(m)_0| < \frac{1}{m+1}$.) Since $|(X_{I'} \setminus \{q'\}) \setminus \pi_{I'}[E]| = |[0, 1] \setminus \pi_0[E]| = \mathfrak{c}$, there is a set Y such that $E \subseteq Y \subseteq X_I$ and $\pi_{I'}[Y] = X_{I'} \setminus \{q'\}$ and $\pi_0|_Y$ is a bijection onto X_0 . We define $f : Y \rightarrow [0, 1]$ by $f(r, x') := f'(x')$ for $(r, x') \in Y \subseteq X_0 \times (X_{I'} \setminus \{q'\}) \subseteq X_0 \times X_{I'} = X_I$.

To see that f is continuous on Y it is enough to note that if $y(\lambda)$ is a net in Y such that $y(\lambda) \rightarrow y = (r, x') \in Y$ then $y(\lambda)_{I'} \rightarrow x'$ with $y(\lambda)_{I'} \in X_{I'} \setminus \{q'\}$ and hence $f(y(\lambda)) = f'(y(\lambda)_{I'}) \rightarrow f'(x') = f(y)$. Clearly f depends on the coordinate 0 $\in I$, i.e., f depends on $J := \{0\} \in [I]^{<\omega}$, and $\pi_0[Y]$ is trivially C -embedded in X_0 since $\pi_0[Y] = X_0$. Also f depends on I' , so there can be no continuous extension \overline{f} of f over X_I : According to the implication (i) \Rightarrow (iv)

of Theorem 4.6 below (with our f' , I' , and \bar{f} playing the role there of f , J , and g , respectively), the existence of such continuous \bar{f} would yield $f' \subseteq \bar{f}_{I'} \in C(X_{I'}, [0, 1])$, a contradiction. \square

Example 2.13. There is a Tychonoff product space $X_I = \prod_{i \in I} X_i$ such that for every nonempty $J \in [I]^{\leq \omega}$ there are a dense subspace $Y \subset X_I$ and a function $f \in C(Y)$ such that

- (a) f depends on J ,
- (b) f_J is continuous,
- (c) f_J does not extend continuously over X_J , and
- (d) f does not extend continuously over X_I .

Proof. Let $\{X_i : i \in I\}$ be a set of metrizable spaces without isolated points, let $J \subseteq I$ satisfy $0 < |J| \leq \omega$, fix $p \in X_J$ and set $D := X_J \setminus \{p\}$. Some $g \in C(D, [0, 1])$ admits no continuous extension over X_J , and then $Y := \pi_J^{-1}(D)$ and $f := g \circ \pi_J|_Y$ are as required (with $f_J = g$ and $\pi_J[Y] = D$). \square

3. EXISTENCE OF CONTINUOUS FACTORIZATIONS

In this section we give some conditions that imply the continuity of functions of the form f_J . We begin with the following observation.

Lemma 3.1. *Let X , Y , and Z be spaces and $f : X \rightarrow Y$, $g : X \rightarrow Z$, and $h : Z \rightarrow Y$ be functions such that $f = h \circ g$. If f is continuous and g is open, then h is continuous.*

Proof. Take U an open set in Y . Then $h^{-1}[U] = g[f^{-1}[U]]$. \square

Theorem 3.2. *Let X_I be a product space, J be a nonempty subset of I , and Y be a nonempty subspace of X_I such that $\pi_J[U \cap Y] = \pi_J[U] \cap \pi_J[Y]$ for every basic open set U of X_I . Let also Z be a space and $f \in C(Y, Z)$ depend on J . Then $f_J : \pi_J[Y] \rightarrow Z$ is continuous.*

Proof. Since $f = f_J \circ \pi_J|_Y$ with f continuous and $\pi_J|_Y$ open, the continuity of f_J follows from Lemma 3.1. \square

Lemma 3.3. *Let X_I be a product space, J be a nonempty proper subset of I , and Y be a subset of X_I . The set $\pi_{I \setminus J}[\pi_J^{-1}(\pi_J(y)) \cap Y]$ is dense in $X_{I \setminus J}$ for every $y \in Y$ if and only if $\pi_J[U] \cap \pi_J[Y] = \pi_J[U \cap Y]$ for every basic open set U in X_I .*

Proof. Let the set $\pi_{I \setminus J}[\pi_J^{-1}(\pi_J(y)) \cap Y]$ be dense in $X_{I \setminus J}$ for every $y \in Y$ and U be a basic open set in X_I . We shall prove that $\pi_J[U] \cap \pi_J[Y] = \pi_J[U \cap Y]$. Let $t \in \pi_J[U] \cap \pi_J[Y]$ and let $x \in Y$ be such that $x_J = t$ and $x_{I \setminus J} \in \pi_{I \setminus J}[U] \cap \pi_{I \setminus J}[Y]$. (Such a point x exists since $\pi_{I \setminus J}[\pi_J^{-1}(t) \cap Y]$ is dense in $X_{I \setminus J}$.) Then $x \in U \cap Y$, hence $t \in \pi_J[U \cap Y]$. Therefore $\pi_J[U] \cap \pi_J[Y] = \pi_J[U \cap Y]$.

Now, let $\pi_J[U] \cap \pi_J[Y] = \pi_J[U \cap Y]$ for every basic open set U in X_I . We shall prove that $\pi_{I \setminus J}[\pi_J^{-1}(\pi_J(y)) \cap Y]$ is dense in $X_{I \setminus J}$ for every $y \in Y$. Let $y \in Y$ and V be a basic open set in $X_{I \setminus J}$. Then $W = \pi_{I \setminus J}^{-1}[V]$ is a basic open set in X_I

and $\pi_J(y) \in \pi_J[W] \cap \pi_J[Y] = \pi_J[W \cap Y]$. Therefore there exists $x \in W \cap Y$ such that $x_J = y_J$ and $x_{I \setminus J} \in \pi_{I \setminus J}[W] = V$, hence $x \in \pi_{I \setminus J}[\pi_J^{-1}(\pi_J(y)) \cap Y] \cap V$. Thus $\pi_{I \setminus J}[\pi_J^{-1}(\pi_J(y)) \cap Y]$ is dense in $X_{I \setminus J}$. \square

Remark 3.4. (a) It is clear that when X_I , Y and J satisfy the (equivalent) conditions of Lemma 3.3, the function $\pi_J|_Y$ is an open map. It is useful to note that the converse can fail. For an example, let X_0 and X_1 be nonempty T_1 -spaces with each $|X_i| > 1$, fix $(x_0, x_1) \in X_0 \times X_1$ and set $Y := ((X_0 \setminus \{x_0\}) \times X_1) \cup \{(x_0, x_1)\}$. Then $\pi_0|_Y$ is an open map, but $\pi_1[\pi_0^{-1}(\pi_0(x_0, x_1)) \cap Y]$, which is the singleton set $\{\pi_1(x_0, x_1)\} = \{x_1\}$, is not dense in X_1 . (Alternatively: with $U := X_0 \times (X_1 \setminus \{x_1\})$ we have $x_0 \in (\pi_0[U] \cap \pi_0[Y]) \setminus \pi_0[U \cap Y]$.)

(b) In a trivial way, using Lemma 3.1, f_J will be continuous provided Y is a nonempty open subspace of a product space X_I and $f \in C(X_I, Z)$ depends on $J \subset I$.

Theorem 3.5. *Let X_I be a product space, J be a nonempty proper subset of I , $\alpha = |J|^+$, and Y be a nonempty subspace of X_I . Let also Z be a space and $f \in C(Y, Z)$ depend on J . Then $f_J : \pi_J[Y] \rightarrow Z$ is a continuous function if*

- (a) $Y = X_I$; or
- (b) $[y \in Y] \Rightarrow \Sigma_\alpha(y) \subseteq Y$; or
- (c) $[J' \subseteq I, |J'| \leq |J|] \Rightarrow \pi_{J'}[Y] = X_{J'}$; or
- (d) $[J' \subseteq I, J' = J \cup F \text{ with } |F| < \omega] \Rightarrow \pi_{J'}[Y] = X_{J'}$; or
- (e) $\pi_J^{-1}[\pi_J[Y]] = Y$; or
- (f) $\pi_{I \setminus J}[\pi_J^{-1}(\pi_J(y)) \cap Y]$ is dense in $X_{I \setminus J}$ for every $y \in Y$; or
- (g) $\pi_J[U] \cap \pi_J[Y] = \pi_J[U \cap Y]$ for every basic open set U in X_I ; or
- (h) $\pi_J[Y] \times \{\bar{y}_{I \setminus J}\} \subseteq Y$ for some $\bar{y} \in Y$.

Proof. Clearly (a) \Rightarrow (b), (b) \Rightarrow (c), and (c) \Rightarrow (d). To see that (d) \Rightarrow (g), let U be a basic open set in X_I and $x_J \in \pi_J[U] \cap \pi_J[Y]$ with $x \in U$, and define $J' := J \cup R(U)$; then since $\pi_{J'}[Y] = X_{J'}$ there is $y \in Y$ such that $y_{J'} = x_{J'}$, so $x_J = y_J \in \pi_J[U \cap Y]$. We conclude that $\pi_J[U] \cap \pi_J[Y] = \pi_J[U \cap Y]$. If (e) holds then $\pi_{I \setminus J}[Y] = X_{I \setminus J}$, so (f) holds, and if (f) holds then (g) holds by Lemma 3.3. Thus by Theorem 3.2 the function f_J is continuous under any of the conditions (a), (b), (c), (d), (e), (f), and (g). If (h) holds and h is the natural homeomorphism from $\pi_J[Y]$ onto $Y' := \pi_J[Y] \times \{\bar{y}_{I \setminus J}\}$, then $f|_{Y'}$ is defined and $f_J = f|_{Y'} \circ h$. \square

Remark 3.6. Theorem 3.5 serves to show that any dense subspace Y of a product space X_I for which some $f \in C(Y, Z)$ depends on $J \subseteq I$ with f_J discontinuous must have properties in common with the spaces from our examples in Section 2. Thus if the answer to Question 1.3 is “No” then the witnessing example must have some of the properties of the dense spaces Y in our examples in Section 2.

4. FACTORIZATIONS AND CONTINUOUS EXTENSIONS

Let $g \in C(X_I, Z)$, Y be a dense subset of X_I , $f := g|_Y$, and $J \subset I$. We know from Example 2.3 and Corollary 2.9 that g may fail to depend on J even if f does depend on J . In this section we give additional conditions sufficient to ensure that this counterintuitive phenomenon cannot occur.

Theorem 4.1. *Let X_I be a product space, J be a nonempty proper subset of I , and Y be a nonempty subspace of X_I such that $\pi_J[Y] \times \{\bar{y}_{I \setminus J}\} \subseteq Y$ for some $\bar{y} \in Y$. Let also Z be a space and $g \in C(X_I, Z)$ be such that the function $f := g|_Y$ depends on J . Then f_J is continuous and has a continuous extension $\bar{f}_J : X_J \rightarrow Z$. Therefore f has a continuous extension $\bar{f} : X_I \rightarrow Z$ that depends on J .*

Proof. The function f_J is continuous according to Theorem 3.5(h). Let $T = X_J \times \{\bar{y}_{I \setminus J}\}$. If $h := g|_T$ then $h : T \rightarrow Z$ is continuous and depends on J . Therefore, according to Theorem 3.5(h) again, the function $h_J : X_J \rightarrow Z$ is continuous. Since $h_J|_{\pi_J[Y]} = f_J$ the function $\bar{f}_J := h_J$ is a continuous extension of f_J . Then the function $\bar{f} := \bar{f}_J \circ \pi_J$ is a continuous extension of f that depends on J , as required. \square

Corollary 4.2. *Let X_I be a product space, J be a nonempty proper subset of I , and Y be a dense subspace of X_I such that $\pi_J[Y] \times \{\bar{y}_{I \setminus J}\} \subseteq Y$ for some $\bar{y} \in Y$. Let also Z be a Hausdorff space and $g \in C(X_I, Z)$ be such that the function $f := g|_Y$ depends on J . Then g depends on J .*

Proof. The domain of agreement of two continuous functions from a fixed space to a Hausdorff space is closed [7, 2.1.9], so with Z Hausdorff we have $\bar{f} = g$ in Theorem 4.1. \square

The following two corollaries are immediate from Corollary 4.2.

Corollary 4.3. *Let $\alpha \geq \omega$, X_I be a product space, Z be a Hausdorff space, $g \in C(X_I, Z)$, J be a nonempty proper subset of I , and $Y \subseteq X_I$ be a Σ_α -space. Then g depends on J if and only if $f := g|_Y$ depends on J .*

Corollary 4.4. *Let X_I be a product space, Z be a Hausdorff space, $g \in C(X_I, Z)$, J be a nonempty proper subset of I , and Y be a dense subspace of X_I such that $Y = \pi_J[Y] \times \pi_{I \setminus J}[Y]$. Then g depends on J if and only if $f := g|_Y$ depends on J .*

The following theorem is a special case of Corollary 4.4. The countable case ($|J| \leq \omega$) is mentioned without proof by N. Noble and M. Ulmer in the proof of Proposition 2.1 in [20].

Theorem 4.5. *Let X_I be a product space, Z be a Hausdorff space, $g \in C(X_I, Z)$, Y_i be a dense subspace of X_i for every $i \in I$, $Y = \prod_{i \in I} Y_i$, and J be a nonempty proper subset of I . Then g depends on J if and only if $f := g|_Y$ depends on J .*

The following theorem contains conditions equivalent to the continuity of a given factorization.

Theorem 4.6. *Let X_I be a product space, Z be a Hausdorff space, $g \in C(X_I, Z)$, Y be a dense subspace of X_I , $f := g|_Y$, and J be a nonempty subset of I . Then the following are equivalent.*

- (i) f depends on J and $f_J : \pi_J[Y] \rightarrow Z$ is continuous.
- (ii) $h := g|_{\pi_J^{-1}[\pi_J[Y]}}$ depends on J .
- (iii) g depends on J .
- (iv) f depends on J , f_J is continuous, and f_J has a continuous extension $\overline{f_J} : X_J \rightarrow Z$.

Proof. (i) \Rightarrow (ii). Let $x, y \in \pi_J^{-1}[\pi_J[Y]]$ and $z \in Y$ be such that $x_J = y_J = z_J$. Then there exists a net $(x_\alpha) \subset Y$ with limit x . Since g is continuous we have $(g(x_\alpha)) \rightarrow g(x)$. Therefore $(f_J((x_\alpha)_J)) \rightarrow g(x)$ for $g(x_\alpha) = f_J((x_\alpha)_J)$ for each α , and since f_J is continuous $(f_J((x_\alpha)_J)) \rightarrow f_J(x_J)$. Thus $g(x) = f_J(x_J) = f_J(z_J) = g(z)$. Similarly we have $g(y) = g(z)$. Therefore $g(x) = g(y)$.

(ii) \Rightarrow (iii). Let $x, y \in X_I$ be such that $x_J = y_J$. If $x_J \in \pi_J[Y]$, then $x, y \in \pi_J^{-1}[\pi_J[Y]]$ and therefore $g(x) = g(y)$ for $g|_{\pi_J^{-1}[\pi_J[Y]}}$ depends on J . Now, let $x_J \notin \pi_J[Y]$; then $x, y \notin \pi_J^{-1}[\pi_J[Y]]$. Since $\pi_J^{-1}[\pi_J[Y]]$ is dense in X_I there exists a net $(x_\alpha) \subset \pi_J^{-1}[\pi_J[Y]]$ such that $(x_\alpha) \rightarrow x$. Then $(g(x_\alpha)) \rightarrow g(x)$ for g is continuous. For each α , we define $w_\alpha(i) = x_\alpha(i)$ for all $i \in J$ and $w_\alpha(i) = y(i)$ for all $i \in I \setminus J$. It is clear that the net $(w_\alpha) \rightarrow y$. Then $(g(w_\alpha)) \rightarrow g(y)$ for g is continuous. Also, $w_\alpha, x_\alpha \in \pi_J^{-1}[\pi_J[Y]]$ and $(w_\alpha)_J = (x_\alpha)_J$ for every α and since $h = g|_{\pi_J^{-1}[\pi_J[Y]}}$ depends on J we have $g(w_\alpha) = g(x_\alpha)$ for every α . Therefore $g(x) = g(y)$. Thus g depends on J .

(iii) \Rightarrow (iv). If g depends on J then g_J is a continuous function which extends f_J .

(iv) \Rightarrow (i). Obvious. □

The following theorem is immediate from Theorem 4.6.

Theorem 4.7. *Let X_I be a product space, Z be a Hausdorff space, $g \in C(X_I, Z)$, and J be a nonempty subset of I . The function g depends on J if and only if there exists a dense subspace Y of X_I such that $f := g|_Y$ depends on J and f_J is continuous.*

As an illustrative application of some of our results we provide now in Theorem 4.10 a proof of a generalization of a classical theorem of A. M. Gleason (see [21, p. 401], [15], or [6]). More general versions of that theorem, with different proofs, can be found in [18], [12], or [14].

Lemma 4.8. *Let $\alpha \geq \omega$, X_I be a product space, $Y \subseteq X_I$ be a dense subset with $|Y| \leq \alpha$, Z be a T_1 -space such that $\chi(Z) \leq \alpha$, and $f \in C(Y, Z)$. Then there exists $J \in [I]^{\leq \alpha}$ such that f depends on J and f_J is continuous.*

Proof. For $y \in Y$ let $\{U(y)_a : a \in A\}$ be a local base at $f(y)$ in Z with $|A| \leq \alpha$. Since the function f is continuous at y , for every $U(y)_a$, $a \in A$, we can find

a basic open neighborhood $V(y)_a$ of y in X_I such that $f[V(y)_a \cap Y] \subset U(y)_a$. Let $J := \cup_{y \in Y} (\cup_{a \in A} R(V(y)_a))$. If two points $x, y \in Y$ are such that $x_J = y_J$ then $f(x) = f(y)$, hence f depends on J .

Now, let $w \in \pi_J[Y]$ and $y \in Y$ be such that $w = y_J$. Let also U be an open neighborhood of $f_J(w) = f(y)$ in Z . Then there exists $b \in A$ such that $f[V(y)_b \cap Y] \subset U$ and since $R(V(y)_b) \subset J$ we have $\pi_J[V(y)_b] \cap \pi_J[Y] = \pi_J[V(y)_b \cap Y]$. Therefore $f_J[\pi_J[V(y)_b] \cap \pi_J[Y]] \subset U$. We conclude that f_J is continuous at w . \square

Lemma 4.8 can be “localized” as follows.

Corollary 4.9. *Let X_I be a product space, Z be a T_1 -space, $Y \subseteq X_I$, J be an infinite subset of I , $f \in C(Y, Z)$ depend on J , $y \in Y$, and $\chi(f(y), Z) \leq |J|$. Then there exists $J_y \subseteq I$ such that $|J_y| = |J|$, f depends on J_y , and f_{J_y} is continuous at y_{J_y} .*

Proof. Let $\{U_a : a \in A\}$ be a local base at $f(y)$ in Z with $|A| \leq |J|$. Since the function f is continuous at y , for every U_a , $a \in A$, we can find a basic open neighborhood V_a of y in X_I such that $f[V_a \cap Y] \subset U_a$. Let $J_y = J \cup (\cup_{a \in A} R(V_a))$. Then $|J_y| = |J|$, f depends on J_y , and f_{J_y} is continuous at y_{J_y} . \square

Theorem 4.10. *Let $\alpha \geq \omega$, $\{X_i : i \in I\}$ be a set of spaces with $d(X_i) \leq \alpha$ for each $i \in I$, Z be a Hausdorff space with $\chi(Z) \leq \alpha$, and $g \in C(X_I, Z)$. Then g depends on $\leq \alpha$ coordinates.*

Proof. We assume without loss of generality, replacing Z by $g[X_I]$ if necessary, that $g : X_I \rightarrow Z$ is a surjection. Since $d(X_i) \leq \alpha$ for each $i \in I$ the cellularity $c(X_I)$ of X_I is $\leq \alpha$ (see [7, 2.3.17], [4, 3.28], [16, 5.6]). Thus, $c(Z) \leq \alpha$ and since $\chi(Z) \leq \alpha$ and Z is Hausdorff we have $|Z| \leq 2^{c(Z)\chi(Z)} = 2^\alpha$ (see [16, 2.15(b)], [11]).

For $z \in Z$ let $\{U(z)_a : a \in A\}$ be a local base at z in Z with $|A| \leq \alpha$. The function g is continuous, hence the set $g^{-1}[U(z)_a]$ is open for every $z \in Z$ and every $a \in A$. Since X_I is a product of spaces with $d(X_i) \leq \alpha$ for each $i \in I$, the closure of every open set depends on $\leq \alpha$ many coordinates (see [4, 10.13]). For $z \in Z$ and $a \in A$, let $J(z, a) \in [I]^{\leq \alpha}$ be a nonempty set such that the set $\overline{g^{-1}[U(z)_a]}$ depends on it and let $K := \cup_{z \in Z} (\cup_{a \in A} J(z, a))$. If $x, y \in X_I$ are such that $x_K = y_K$ and $g(x) = z$ then $y \in \overline{g^{-1}[U(z)_a]} \subseteq g^{-1}[\overline{U(z)_a}]$ for every $a \in A$ and since Z is Hausdorff $\{z\} = \cap_{a \in A} \overline{U(z)_a}$, thus $y \in g^{-1}(z)$. Therefore $g(x) = g(y)$, hence g depends on K . Thus, $g_K : X_K \rightarrow Z$ is continuous and since $|K| \leq 2^\alpha$ it follows from Hewitt-Marczewski-Pondiczery theorem (see [7, 2.3.15]) that X_K contains a dense set $Y \in [X_K]^{\leq \alpha}$. Then by Lemma 4.8 (with X_K now in place of X_I) there exists $J \in [K]^{\leq \alpha}$ such that g_K depends on J . Therefore g depends on J . \square

Remark 4.11. As it is clear from the proof of Theorem 4.10, the hypothesis $\chi(Z) \leq \alpha$ there may be relaxed to the condition $2^{\chi(Z)} \leq 2^\alpha$.

Since $\chi(Z) \leq w(Z)$ for every space Z , Theorem 4.10 gives a proof of Theorem 10.14 in [4]:

Corollary 4.12. *Let $\alpha \geq \omega$, $\{X_i : i \in I\}$ be a set of spaces with $d(X_i) \leq \alpha$ for $i \in I$, Z be a Hausdorff space with $w(Z) \leq \alpha$, and $g \in C(X_I, Z)$. Then g depends on $\leq \alpha$ coordinates.*

The foregoing results afford several conditions which ensure that (under suitable hypotheses) a function $g : X_I \rightarrow Z$ depends on a set $J \subseteq I$ iff the restricted function $g|_Y$ does so. We record two instances of particular interest.

Theorem 4.13. *Let X_I be a product space, Z be a Hausdorff space, $g \in C(X_I, Z)$, J be a nonempty proper subset of I , and Y be a dense subset of X_I . If either*

- (a) $[J' \subseteq I, |J'| \leq |J|] \Rightarrow \pi_{J'}[Y] = X_{J'}$, or
- (b) $\pi_{I \setminus J}[\pi_J^{-1}(\pi_J(y)) \cap Y]$ is dense in $X_{I \setminus J}$ for every $y \in Y$,

then g depends on J iff $f := g|_Y$ depends on J .

Proof. Surely if g depends on J then $f = g|_Y$ depends on J . For the reverse implications, it is enough to note from Theorem 3.5 that f_J is continuous, so the implication (i) \Rightarrow (iii) of Theorem 4.6 applies. \square

5. ON THE COMFORT-NEGREPONTIS QUESTION (QUESTION 1.3)

In this section we give some sufficient conditions for a positive answer to the Comfort–Negrepontis question. We begin with a workable equivalent condition which in particular cases is readily verified (or, as in Section 2, refuted).

Theorem 5.1. *Let $\alpha \geq \omega$, X_I be a product space, Z be a space, $Y \subseteq X_I$ be a dense subspace such that $\pi_J[Y]$ is $C(Z)$ -embedded in X_J for every nonempty $J \in [I]^{<\alpha}$, and $f \in C(Y, Z)$. The function f has a continuous extension $\bar{f} : X_I \rightarrow Z$ that depends on $< \alpha$ coordinates if and only if there exists a nonempty $J \in [I]^{<\alpha}$ such that f depends on J and f_J is continuous.*

Proof. If $\bar{f} \in C(X_I, Z)$ depends on some nonempty $J \in [I]^{<\alpha}$ then \bar{f}_J is continuous (see Corollary 3.2). Therefore f_J is continuous.

If f depends on J and f_J is continuous, then since $\pi_J[Y]$ is $C(Z)$ -embedded in X_J there is \bar{f}_J such that $f_J \subseteq \bar{f}_J \in C(X_J, Z)$. Then $\bar{f} := \bar{f}_J \circ \pi_J$ is a continuous extension of f and \bar{f} depends on J . \square

It is easily seen from the proof of the above theorem that if we need to extend a particular function f that depends on a given nonempty set $J \subseteq I$ then we do not require the strong hypothesis that $\pi_J[Y]$ is $C(Z)$ -embedded in X_J . We need only that f_J is continuous and that it extends to continuous $\bar{f}_J : X_J \rightarrow Z$. Therefore Theorem 5.1 could be restated as follows.

Theorem 5.2. *Let X_I be a product space, Z be a space, J be a nonempty subset of I , $Y \subseteq X_I$ be a dense subspace, and $f \in C(Y, Z)$ depend on J . The function f has a continuous extension $\bar{f} : X_I \rightarrow Z$ that depends on J if and only if f_J is continuous and has a continuous extension $\bar{f}_J : X_J \rightarrow Z$.*

The next theorem generalizes Theorem 1.4 and gives conditions sufficient that the answer to Question 1.3 is in the affirmative.

Theorem 5.3. *Let X_I be a product space, J be a nonempty proper subset of I , $\alpha = |J|^+$, and Y be a nonempty subspace of X_I . Let also Z be a space and $f \in C(Y, Z)$ depend on J . If*

- (a) $Y = X_I$; or
- (b) $[y \in Y] \Rightarrow \Sigma_\alpha(y) \subseteq Y$; or
- (c) $[J' \subseteq I, |J'| \leq |J|] \Rightarrow \pi_{J'}[Y] = X_{J'}$; or
- (d) $[J' \subseteq I, J' = J \cup F \text{ with } |F| < \omega] \Rightarrow \pi_{J'}[Y] = X_{J'}$; or
- (e) $\pi_J^{-1}[\pi_J[Y]] = Y$ and $\pi_J[Y]$ is $C(Z)$ -embedded in X_J ; or
- (f) $\pi_{I \setminus J}[\pi_J^{-1}(\pi_J(y)) \cap Y]$ is dense in $X_{I \setminus J}$ for every $y \in Y$ and $\pi_J[Y]$ is $C(Z)$ -embedded in X_J ; or
- (g) $\pi_J[U \cap \pi_J[Y]] = \pi_J[U \cap Y]$ for every basic open set U in X_I and $\pi_J[Y]$ is $C(Z)$ -embedded in X_J ; or
- (h) $\pi_J[Y] \times \{\bar{y}_{I \setminus J}\} \subseteq Y$ for some $\bar{y} \in Y$ and $\pi_J[Y]$ is $C(Z)$ -embedded in X_J

then f has a continuous extension $\bar{f} : X_I \rightarrow Z$ that depends on J .

Proof. As in Theorem 3.5, clearly (a) \Rightarrow (b), (b) \Rightarrow (c), (c) \Rightarrow (d), and (d) \Rightarrow (g). Also (e) \Rightarrow (f) and (f) \Rightarrow (g). If (g) holds then f_J is continuous by Theorem 3.5; thus $\bar{f} := \bar{f}_J \circ \pi_J|_Y$ is a continuous extension of f that depends on J , so under any of the conditions (a), (b), (c), (d), (e), (f), and (g) f has a continuous extension that depends on J . Similarly, if (h) holds then f_J is continuous by Theorem 3.5; thus f has a continuous extension that depends on J . \square

A partial answer, in the positive, to Question 1.3 is given in Theorem 5.6.

Lemma 5.4. *Let X be a space, $p \in X$, and (W_n) be a sequence of open subsets of X such that $W_{n+1} \subseteq \overline{W_{n+1}}^X \subseteq W_n$ and $\bigcap_n W_n = \{p\}$. Then (W_n) is locally finite at each point of $X \setminus \{p\}$.*

Proof. We are to show for $p \neq x \in X$ that some neighborhood U of x meets W_n for only finitely many n . Given x , it is enough to choose k such that $x \notin \overline{W_k}^X$ and to set $U := X \setminus \overline{W_k}^X$. \square

We say as usual that a subset Y of a space X is *sequentially closed* if $[y_n \in Y \text{ and } y_n \rightarrow p \in X] \Rightarrow p \in Y$, and X is a *sequential space* if every sequentially closed subset of X is closed.

Theorem 5.5. *Let X be a Tychonoff space with countable pseudocharacter and let Y be a C^* -embedded subset of X . Then Y is sequentially closed in X .*

Proof. Suppose there is a sequence $y_n \in Y$ such that $y_n \rightarrow p \in X \setminus Y$, and let (U_n) be a (countable) local pseudobase at p in X .

Let $V_{n_0} := U_0$ and choose $y_{n_0} \in V_{n_0}$. Suppose that V_{n_k} and y_{n_k} have been defined. Let $n_{k+1} > n_k$ be such that $y_{n_{k+1}} \in V_{n_{k+1}}$, where $V_{n_{k+1}}$ is a

neighborhood in X of p such that $V_{n_{k+1}} \subseteq \overline{V_{n_{k+1}}}^X \subseteq V_{n_k} \cap U_{n_k}$ and $y_{n_k} \notin \overline{V_{n_{k+1}}}^X$. With the sequences V_{n_k} and y_{n_k} so defined, choose a function $f_k \in C(X, [0, 1])$ such that $f_k \equiv 0$ for k even, and for odd k , using the Tychonoff property of X , such that $0 \leq f_k \leq 1$, $f_k(y_{n_k}) = 1$, and $f_k \equiv 0$ on $(X \setminus V_{n_k}) \cup \overline{V_{n_{k+1}}}^X$. Since the sequence $V_{n_k} \setminus \overline{V_{n_{k+1}}}^X$ is pairwise disjoint, the function $f := \sum_k f_k : X \rightarrow [0, 1]$ is well-defined. According to Lemma 5.4, the sequence (V_{n_k}) is locally finite at each point of $X \setminus \{p\}$, so the sequence $((V_{n_k} \setminus \overline{V_{n_{k+1}}}^X) \cap Y)$ is locally finite in Y ; thus $g := f|_Y \in C^*(Y)$. For each neighborhood U of p in X there is k such that $y_{n_{2k}}, y_{n_{2k+1}} \in U$. We have $y_{n_{2k+1}} \in U \cap Y$ with $g(y_{n_{2k+1}}) = 1$ and $y_{n_{2k}} \in V_{n_{2k}} \cap Y$ with $g(y_{n_{2k}}) = 0$. We conclude that g does not extend continuously to p , a contradiction. \square

Theorem 5.6. *Let $\alpha \leq \omega_1$ and \mathcal{S} be a class of Tychonoff spaces such that if $S_n \in \mathcal{S}$ ($n < \omega$, repetitions permitted) then $\prod_{n < \omega} S_n \in \mathcal{S}$ and $\prod_{n < \omega} S_n$ is a sequential space. Let also $\{X_i : i \in I\} \subseteq \mathcal{S}$ with $\psi(X_i) \leq \omega$ for each $i \in I$, and Y be a dense subspace of X_I such that $\pi_J[Y]$ is C^* -embedded in X_J for every nonempty $J \in [I]^{<\alpha}$. If $f \in C^*(Y)$ and there is $J \in [I]^{<\alpha}$ such that f depends on J , then f extends continuously over X_I .*

Proof. By virtue of Lemma 1.2 (taking $\kappa = \omega$ and $Z = \mathbb{R}$ there), it suffices to prove that $\pi_J[Y] = X_J$ for all $J \in [I]^{<\alpha}$. Take $J \in [I]^{<\alpha}$. From $\psi(X_i) \leq \omega$ (all $i \in J$) and $|J| \leq \omega$ it follows easily that $\psi(X_J) \leq \omega$, so $\pi_J[Y]$ is sequentially closed in X_J by Theorem 5.5 and hence closed in X_J , so $\pi_J[Y] = X_J$. \square

Remark 5.7. The class of all Tychonoff, first countable spaces of course satisfies the hypotheses required of \mathcal{S} in the statement of Theorem 5.6, but other less familiar classes of spaces do so as well. For example, one may take for \mathcal{S} the class of all Tychonoff, bi-sequential spaces (see [17, Definition 3.D.1]) with countable pseudocharacter but not first countable. (It is known that every bi-sequential space is sequential ([17]); the class of bi-sequential spaces is closed under countable products ([17]); there exists a bi-sequential Tychonoff space without countable pseudocharacter ([17, Example 10.4]); and there exists a bi-sequential Tychonoff space with countable pseudocharacter which is not first countable (see Example 5.8 below).)

We are indebted to Gary Gruenhagen for the following example.

Example 5.8. Let $K := [0, 1]$ be the interval $[0, 1]$ with the usual topology. For every rational $q \in K$, choose the constant sequence $S_q := \{q\}$ and for every irrational $r \in K$, choose a sequence S_r of rational numbers in K converging to r . Define $S := \cup_{x \in K} S_x$ and $X := S \cup \{\infty\}$. Let \mathcal{T} be the smallest topology on X such that:

(1) For every $q \in S$, $\{q\}$ is \mathcal{T} -open in X ; and

(2) For every finite subset F of S and every finite subset G of $[0, 1]$ the set $N(\infty, F, G) := \{\infty\} \cup [S \setminus (F \cup \cup_{x \in G} S_x)]$ is a \mathcal{T} -open neighborhood of ∞ .

It follows from [10, Proposition 3.2] that X is bi-sequential. The family $\{N(\infty, \emptyset, \{q\}) : q \in S\}$ is countable and $\{\infty\} = \cap_{q \in S} N(\infty, \emptyset, \{q\})$. Hence

$\psi(X) = \omega$. Now take a countable family $\{N(\infty, F_n, G_n) : n < \omega\}$ of neighborhoods of ∞ . Choose an irrational $r \in K$ not in $\cup_{n < \omega} G_n$. Then $N(\infty, \emptyset, \{r\})$ contains no $N(\infty, F_n, G_n)$, $n < \omega$. Therefore X is not first countable. Since each of the \mathcal{T} -basic open sets specified in (1) and (2) is also \mathcal{T} -closed, the Hausdorff space (X, \mathcal{T}) is a Tychonoff space.

6. CONCLUDING REMARKS

We remark that most of the results of the foregoing sections admit natural generalizations to the κ -box topology. Since these are routine and their formulation adds little to our broad understanding, we leave the details to the interested reader. We note explicitly, however, that Question 1.3 remains open even in the case $\kappa = \omega$.

We anticipate that Question 1.3 has a negative answer. When seeking a positive response, however, we have been drawn to the following question, which is closely related to Theorem 2.8. Clearly, a positive answer to 6.1 will respond positively also to 1.3 (in the case that Z is a Hausdorff space).

Problem 6.1. *Let $\alpha \geq \omega$, X_I be a product space, Z be a space, and Y be a dense subspace of X_I such that $\pi_J[Y]$ is $C(Z)$ -embedded in X_J for every nonempty $J \in [I]^{<\alpha}$. If $f \in C(Y, Z)$ depends on some nonempty $J' \in [I]^{<\alpha}$, must there be a nonempty $J \in [I]^{<\alpha}$ such that f depends on J and f_J is continuous?*

Returning finally to Question 1.3, we remark that if a counterexample exists then there will be a product space X_I , a cardinal number $\alpha \geq \omega$, a dense subspace Y of X_I , a space Z , and a function $f \in C(Y, Z)$ such that $\pi_J[Y]$ is $C(Z)$ -embedded in X_J for every $J \in [I]^{<\alpha}$, f depends on some nonempty $J \in [I]^{<\alpha}$, and f does not extend continuously over X_I . In particular, then, f_J must be discontinuous for each $J \in [I]^{<\alpha}$ on which f depends.

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