ON TWO CONJECTURES ABOUT FAULTY HYPERCUBES

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Abstract. Let $\mathcal{F}$ be a set of vertices in the binary hypercube $\mathcal{Q}_n$. A set $\mathcal{F}'$ of vertices in $\mathcal{Q}_n$ is a *disjoint mirror twin* of $\mathcal{F}$ if $\mathcal{F}'$ has the same cardinality as $\mathcal{F}$, $\mathcal{F} \cap \mathcal{F}' = \emptyset$, and the number of even vertices in $\mathcal{F}$ is equal to the number of odd vertices in $\mathcal{F}'$.

In this paper the following two results are proved. The first one is a weaker version of Locke's conjecture: The binary hypercube $\mathcal{Q}_n$ with f deleted vertices of each parity is Hamiltonian if $n \geq f+2$. The second one is a weaker version of Castañeda–Gotchev's conjecture: If $n \geq 3$, $0 \leq f \leq n-3$, and $\mathcal{F}$ is a set of f even and f odd vertices of $\mathcal{Q}_n$ then for every pair of vertices $u, v \in \mathcal{V}(\mathcal{Q}_n - \mathcal{F})$ with opposite parity there exists a Hamiltonian path in $\mathcal{Q}_n - \mathcal{F}$ from u to v.

Theorem 1. Let $n \geq 2$ and $\mathcal{F}$ be a set of vertices in $\mathcal{Q}_n$ of cardinality $0 \leq f \leq n-2$. Then there exists a disjoint mirror twin $\mathcal{F}'$ of $\mathcal{F}$ such that the graph $\mathcal{Q}_n - (\mathcal{F} \cup \mathcal{F}')$ is Hamiltonian.

Theorem 2. Let $n \geq 3$ and $\mathcal{F}$ be a set of vertices in $\mathcal{Q}_n$ of cardinality $0 \leq f \leq n-3$. Then for every pair of vertices $u, v \in \mathcal{V}(\mathcal{Q}_n - \mathcal{F})$ with opposite parity there exists a disjoint mirror twin $\mathcal{F}'$ of $\mathcal{F}$, with $\{u, v\} \cap \mathcal{F}' = \emptyset$, such that there exists a Hamiltonian path for $\mathcal{Q}_n - (\mathcal{F} \cup \mathcal{F}')$ from u to v.

1. Introduction and definitions

The n-dimensional hypercube $\mathcal{Q}_n$ is the graph whose vertices are the binary sequences of length n and whose edges are pairs of binary sequences that differ in exactly one position. A given vertex is called *even* if it has an even number of 1's in its binary representation; otherwise the vertex is called *odd*.

Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a set of vertices $S \subset \mathcal{V}(\mathcal{G})$, the subgraph of $\mathcal{G}$ *induced* by S is the graph whose vertex set is S and whose edge set consists of all edges in $\mathcal{E}(G)$ whose endpoints are both in S. We denote by $\mathcal{G} - S$ the subgraph of $\mathcal{G}$ induced by the vertex set $\mathcal{V}(\mathcal{G}) \setminus S$.

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Throughout this paper, we shall say that a vertex u is a *neighbor* of a vertex v if u and v are adjacent in $\mathcal{Q}_n$.

Locke's conjecture [L] states that the binary hypercube $\mathcal{Q}_n$ with f deleted vertices of each parity is Hamiltonian if $n \geq f+2$. In 2003, S. C. Locke and R. Stong published in *The American Mathematical Monthly* a proof of that conjecture for the case $f = 1$ [LS]. In [CG1] Castañeda and Gotchev proved that conjecture for every $f \leq 4$ (for related results see [CG2]).

In the same paper the following conjecture was also stated (see [CG1, 6.3]) and verified for each $f \leq 2$ and each $n \geq f+3$: Let $n \geq 3$ and $0 \leq f \leq n-3$. If $\mathcal{F}$ is a set of f even and f odd vertices of $\mathcal{Q}_n$ then for every pair of vertices $u, v \in \mathcal{V}(\mathcal{Q}_n - \mathcal{F})$ with opposite parity there exists a Hamiltonian path for $\mathcal{Q}_n - \mathcal{F}$ from u to v.

Here we formulate and prove weaker forms of the above two conjectures. For that we need the following definition.

Definition 1.1. *Let $\mathcal{F}$ be a set of vertices in $\mathcal{Q}_n$. We shall say that a set $\mathcal{F}'$ of vertices in $\mathcal{Q}_n$ is a* mirror twin *of $\mathcal{F}$ if $\mathcal{F}'$ has the same cardinality as $\mathcal{F}$, the number of even vertices in $\mathcal{F}$ is equal to the number of odd vertices in $\mathcal{F}'$, and there is the same number of even and odd vertices in $\mathcal{F} \cap \mathcal{F}'$. In the case when $\mathcal{F} \cap \mathcal{F}' = \emptyset$ we shall call $\mathcal{F}'$ a* disjoint mirror twin *of $\mathcal{F}$.*

Using the above defined terms Locke's conjecture could be restated as follows.

Conjecture 1.2 (Locke)**.** *Let $n \geq 2$ and $\mathcal{F}$ be a set of vertices in $\mathcal{Q}_n$ of cardinality $0 \leq f \leq n-2$. Then for every disjoint mirror twin $\mathcal{F}'$ of $\mathcal{F}$ the graph $\mathcal{Q}_n - (\mathcal{F} \cup \mathcal{F}')$ is Hamiltonian.*

The following conjecture sounds more general than Locke's conjecture but is actually equivalent to it.

Conjecture 1.3. *Let $n \geq 2$ and $\mathcal{F}$ be a set of vertices in $\mathcal{Q}_n$ of cardinality $0 \leq f \leq n-2$. Then for every mirror twin $\mathcal{F}'$ of $\mathcal{F}$ the graph $\mathcal{Q}_n - (\mathcal{F} \cup \mathcal{F}')$ is Hamiltonian.*

Using the same terminology Castañeda–Gotchev's conjecture could be reformulated as follows.

Conjecture 1.4. *Let $n \geq 3$ and $\mathcal{F}$ be a set of vertices in $\mathcal{Q}_n$ of cardinality $0 \leq f \leq n-3$. Then for every pair of vertices $u, v \in \mathcal{V}(\mathcal{Q}_n - \mathcal{F})$ with opposite parity and every mirror twin $\mathcal{F}'$ of $\mathcal{F}$, with $\{u, v\} \cap \mathcal{F}' = \emptyset$, there exists a Hamiltonian path for $\mathcal{Q}_n - (\mathcal{F} \cup \mathcal{F}')$ from u to v.*

It is easy to verify that the conjecture obtained from Conjecture 1.4 by replacing "every mirror twin" with "every disjoint mirror twin" is equivalent to Conjecture 1.4.

In this paper we prove the following theorem which is weaker than Locke's conjecture since it follows from it but does not imply it.

Theorem 1.5. *Let $n \geq 2$, $0 \leq f \leq n-2$, and $\mathcal{F}$ be a set of vertices in $\mathcal{Q}_n$ of cardinality f. Then there exists a disjoint mirror twin $\mathcal{F}'$ of $\mathcal{F}$ such that the graph $\mathcal{Q}_n - (\mathcal{F} \cup \mathcal{F}')$ is Hamiltonian.*

Since Theorem 1.5 follows from Locke's conjecture, which is true for each $f \leq 4$ and each $n \geq f+2$, we have to prove Theorem 1.5 only for $n \geq 7$ and $5 \leq f \leq n-2$. In its proof we use the following theorem, which is a weaker form of Castañeda–Gotchev's conjecture and therefore it is known to be true for each $f \leq 2$ and each $n \geq f+3$.

Theorem 1.6. *Let $n \geq 3$, $0 \leq f \leq n-3$, and $\mathcal{F}$ be a set of vertices in $\mathcal{Q}_n$ of cardinality f. Then for every pair of vertices $u, v \in \mathcal{V}(\mathcal{Q}_n - \mathcal{F})$ with opposite parity there exists a disjoint mirror twin $\mathcal{F}'$ of $\mathcal{F}$, with $\{u, v\} \cap \mathcal{F}' = \emptyset$, such that there exists a Hamiltonian path for $\mathcal{Q}_n - (\mathcal{F} \cup \mathcal{F}')$ from u to v.*

2. Proof of Theorem 1.6

As it is mentioned above, Theorem 1.6 is true for each $f \leq 2$ and each $n \geq f+3$. We continue the proof by induction on f. Let $f_0 \geq 3$ and suppose that the claim is true for each $f < f_0$ and each $n \geq f+3$. We shall prove the claim for $f = f_0$ and every $n \geq f+3$. Notice that since $f \geq 3$ we have $n \geq 6$.

Let $f = f_0$ and $n \geq f+3$. Let also $\mathcal{F}$ be a set of vertices in $\mathcal{Q}_n$ of cardinality f and $u, v \in \mathcal{V}(\mathcal{Q}_n - \mathcal{F})$ be a pair of vertices with opposite parity.

Since $f \geq 3$, there must be a coordinate where not all vertices of $\mathcal{F}$ agree. Suppose, without loss of generality, that it is the first coordinate. For $i \in \{0,1\}$, let $\mathcal{V}_i$ be the subset of $\mathcal{V}(\mathcal{Q}_n)$ consisting of the vertices whose first coordinate is i. Then $1 \leq |\mathcal{V}_i \cap \mathcal{F}| \leq f-1$ for $i \in \{0,1\}$, but also, since $f \geq 3$, one of $|\mathcal{V}_0 \cap \mathcal{F}|$ and $|\mathcal{V}_1 \cap \mathcal{F}|$ must be at least two. Suppose, without loss of generality, that $|\mathcal{V}_0 \cap \mathcal{F}| \geq 2$. Then there must be a coordinate, other than the first, where not all vertices of $\mathcal{V}_0 \cap \mathcal{F}$ agree. Without loss of generality, suppose that it is the second coordinate. For $i, j \in \{0,1\}$, let $\mathcal{V}_{ij}$ be the subset of $\mathcal{V}(\mathcal{Q}_n)$ consisting of the vertices whose first two coordinates are ij, and let $\mathcal{F}_{ij} = \mathcal{V}_{ij} \cap \mathcal{F}$. Since $\mathcal{V}_{00}, \mathcal{V}_{01}$ and $\mathcal{V}_1$ all contain at least one vertex of $\mathcal{F}$, we have that $|\mathcal{F}_{ij}| \leq f-2$ for all $i, j \in \{0,1\}$.

Now, for $i, j \in \{0, 1\}$, let $\mathcal{Q}_n^{ij}$ be the subgraph of $\mathcal{Q}_n$ induced by $\mathcal{V}_{ij}$. Then each $\mathcal{Q}_n^{ij}$ is an $n-2$-dimensional hypercube, and since $|\mathcal{F}_{ij}|+3 \leq (f-2)+3 = f+1 \leq n-2$, we can apply the induction hypothesis for each pair of vertices of opposite parity in $\mathcal{Q}_n^{ij} - \mathcal{F}_{ij}$.

Case 1. $u, v \in \mathcal{V}(\mathcal{Q}_n^{00} - \mathcal{F}_{00})$.

According to the induction hypothesis, applied for $u, v \in \mathcal{V}(\mathcal{Q}_n^{00} - \mathcal{F}_{00})$, there exists a disjoint mirror twin $\mathcal{F}'_{00}$ of $\mathcal{F}_{00}$, with $\{u, v\} \cap \mathcal{F}'_{00} = \emptyset$, such that there exists a Hamiltonian path γ for $\mathcal{Q}_n^{00} - (\mathcal{F}_{00} \cup \mathcal{F}'_{00})$ from u to v. The length of γ is $2^{n-2} - 2|\mathcal{F}_{00}| - 1$. Since $n \geq 6$ we have that

$$2^{n-2} - 2|\mathcal{F}_{00}| - 1 \geq 2^{n-2} - 2(n-5) - 1 > 2(n-3) \geq 2f.$$

Therefore if we project all vertices from $\mathcal{F}$ onto $\mathcal{Q}_n^{00}$ (parallel to the first two coordinates), there will be at least two consecutive vertices u_{00} and v_{00} in γ that are not projections of any vertices from $\mathcal{F}$. Without loss of generality, we can assume that v_{00} is closer to u in γ than u_{00}. Denote by γ_1 the path defined by γ from u to v_{00} and by γ_2 the path from u_{00} to v. Denote also the neighbors of u_{00} and v_{00} in $\mathcal{V}_{01}$ and $\mathcal{V}_{10}$, respectively, by v_{01}, u_{01} and v_{10}, u_{10}, and their neighbors in $\mathcal{V}_{11}$ by u_{11}, v_{11}.

Clearly, none of the vertices u_{01}, v_{01}, u_{10}, v_{10}, u_{11}, or v_{11} belong to $\mathcal{F}$. Therefore we can apply the induction hypothesis for $u_{01}, v_{01} \in \mathcal{V}(\mathcal{Q}_n^{01} - \mathcal{F}_{01})$ to find a disjoint mirror twin $\mathcal{F}'_{01}$ of $\mathcal{F}_{01}$ in $\mathcal{Q}_n^{01}$ such that $\{u_{01}, v_{01}\} \cap \mathcal{F}'_{01} = \emptyset$ and a Hamiltonian path γ_3 for $\mathcal{Q}_n^{01} - (\mathcal{F}_{01} \cup \mathcal{F}'_{01})$ from u_{01} to v_{01}. In the same way, applying the induction hypothesis for $u_{10}, v_{10} \in \mathcal{V}(\mathcal{Q}_n^{10} - \mathcal{F}_{10})$ we can find a disjoint mirror twin $\mathcal{F}'_{10}$ of $\mathcal{F}_{10}$ in $\mathcal{Q}_n^{10}$ such that $\{u_{10}, v_{10}\} \cap \mathcal{F}'_{10} = \emptyset$ and a Hamiltonian path γ_4 for $\mathcal{Q}_n^{10} - (\mathcal{F}_{10} \cup \mathcal{F}'_{10})$ from u_{10} to v_{10}. Finally, applying the induction hypothesis for $u_{11}, v_{11} \in \mathcal{V}(\mathcal{Q}_n^{11} - \mathcal{F}_{11})$ we can find a disjoint mirror twin $\mathcal{F}'_{11}$ of $\mathcal{F}_{11}$ in $\mathcal{Q}_n^{11}$ such that $\{u_{11}, v_{11}\} \cap \mathcal{F}'_{11} = \emptyset$ and a Hamiltonian path γ_5 for $\mathcal{Q}_n^{11} - (\mathcal{F}_{11} \cup \mathcal{F}'_{11})$ from u_{11} to v_{11}. Then $\mathcal{F}' = \mathcal{F}'_{00} \cup \mathcal{F}'_{01} \cup \mathcal{F}'_{10} \cup \mathcal{F}'_{11}$ is a disjoint mirror twin of $\mathcal{F}$ and the path

$$u \xrightarrow{\gamma_1} v_{00} \to u_{01} \xrightarrow{\gamma_3} v_{01} \to u_{11} \xrightarrow{\gamma_5} v_{11} \to u_{10} \xrightarrow{\gamma_4} v_{10} \to u_{00} \xrightarrow{\gamma_2} v$$

is the required Hamiltonian path from u to v for $\mathcal{Q}_n - (\mathcal{F} \cup \mathcal{F}')$.

Case 2. $u \in \mathcal{V}(\mathcal{Q}_n^{00} - \mathcal{F}_{00})$, $v \in \mathcal{V}(\mathcal{Q}_n^{10} - \mathcal{F}_{10})$.

Since $|\mathcal{F}| = f \leq n-3$ and $\mathcal{Q}_n^{00}$ has dimension $n-2$, if we project all vertices from $\mathcal{F}$ onto $\mathcal{Q}_n^{00}$ (parallel to the first two coordinates), there will be at least one neighbor $v_{00} \in \mathcal{V}_{00}$ of u which will not be a projection of any vertex from $\mathcal{F}$. Denote the neighbor of v_{00} in $\mathcal{V}_{01}$ by u_{01}, in $\mathcal{V}_{10}$ by u_{10}, and their neighbor in $\mathcal{V}_{11}$ by v_{11}. For the same reason, if we project all vertices from $\mathcal{F}$ onto $\mathcal{Q}_n^{01}$, there will be at least one neighbor $v_{01} \in \mathcal{V}_{01}$ of u_{01}, which will not be a projection of any vertex from $\mathcal{F}$. Denote the neighbor of v_{01} in $\mathcal{V}_{11}$ by u_{11}.

Clearly, none of the vertices v_{00}, u_{01}, v_{01}, u_{11}, v_{11}, or u_{10} belong to $\mathcal{F}$. Therefore we can apply the induction hypothesis for $u, v_{00} \in \mathcal{V}(\mathcal{Q}_n^{00} - \mathcal{F}_{00})$ to find a disjoint mirror twin $\mathcal{F}'_{00}$ of $\mathcal{F}_{00}$ in $\mathcal{Q}_n^{00}$ such that $\{u, v_{00}\} \cap \mathcal{F}'_{00} = \emptyset$ and a Hamiltonian path γ_1 for $\mathcal{Q}_n^{00} - (\mathcal{F}_{00} \cup \mathcal{F}'_{00})$ from u to v_{01}. In the same way, applying the induction hypothesis for $u_{01}, v_{01} \in \mathcal{V}(\mathcal{Q}_n^{01} - \mathcal{F}_{01})$ we can find a disjoint mirror twin $\mathcal{F}'_{01}$ of $\mathcal{F}_{01}$ in $\mathcal{Q}_n^{01}$ such that $\{u_{01}, v_{01}\} \cap \mathcal{F}'_{01} = \emptyset$ and a Hamiltonian path γ_2 for $\mathcal{Q}_n^{01} - (\mathcal{F}_{01} \cup \mathcal{F}'_{01})$ from u_{01} to v_{01}. Also, applying the induction hypothesis for $u_{11}, v_{11} \in \mathcal{V}(\mathcal{Q}_n^{11} - \mathcal{F}_{11})$ we can find a disjoint mirror twin $\mathcal{F}'_{11}$ of $\mathcal{F}_{11}$ in $\mathcal{Q}_n^{11}$ such that $\{u_{11}, v_{11}\} \cap \mathcal{F}'_{11} = \emptyset$ and a Hamiltonian path γ_3 for $\mathcal{Q}_n^{11} - (\mathcal{F}_{11} \cup \mathcal{F}'_{11})$ from u_{11} to v_{11}. Finally, applying the induction hypothesis for $u_{10}, v \in \mathcal{V}(\mathcal{Q}_n^{10} - \mathcal{F}_{10})$ we can find a disjoint mirror twin $\mathcal{F}'_{10}$ of $\mathcal{F}_{10}$ in $\mathcal{Q}_n^{10}$ such that $\{u_{10}, v\} \cap \mathcal{F}'_{10} = \emptyset$ and a Hamiltonian path γ_4 for $\mathcal{Q}_n^{10} - (\mathcal{F}_{10} \cup \mathcal{F}'_{10})$ from u_{10} to v. Then $\mathcal{F}' = \mathcal{F}'_{00} \cup \mathcal{F}'_{01} \cup \mathcal{F}'_{10} \cup \mathcal{F}'_{11}$ is a disjoint mirror twin of $\mathcal{F}$ and the path

$$u \xrightarrow{\gamma_1} v_{00} \to u_{01} \xrightarrow{\gamma_2} v_{01} \to u_{11} \xrightarrow{\gamma_3} v_{11} \to u_{10} \xrightarrow{\gamma_4} v$$

is the required path from u to v for $\mathcal{Q}_n - (\mathcal{F} \cup \mathcal{F}')$.

Case 3. $u \in \mathcal{V}(\mathcal{Q}_n^{00} - \mathcal{F}_{00})$, $v \in \mathcal{V}(\mathcal{Q}_n^{11} - \mathcal{F}_{11})$.

Since $|\mathcal{F}| = f \leq n - 3$ and $\mathcal{Q}_n^{00}$ has dimension $n - 2$, if we project all vertices from $\mathcal{F}$ onto $\mathcal{Q}_n^{00}$ (parallel to the first two coordinates), there will be at least one neighbor $v_{00} \in \mathcal{V}_{00}$ of u which will not be a projection of any vertex from $\mathcal{F}$. Denote the neighbor of v_{00} in $\mathcal{V}_{01}$ by u_{01}. For the same reason, if we project all vertices from $\mathcal{F}$ onto $\mathcal{Q}_n^{01}$ (again parallel to the first two coordinates), there will be at least one neighbor $v_{01} \in \mathcal{V}_{01}$ of u_{01}, which will not be a projection of any vertex from $\mathcal{F}$. Denote the neighbor of v_{01} in $\mathcal{V}_{11}$ by u_{11}.

Clearly, none of the vertices v_{00}, u_{01}, v_{01}, or u_{11} belong to $\mathcal{F}$. Therefore we can apply the induction hypothesis for $u, v_{00} \in \mathcal{V}(\mathcal{Q}_n^{00} - \mathcal{F}_{00})$ to find a disjoint mirror twin $\mathcal{F}'_{00}$ of $\mathcal{F}_{00}$ in $\mathcal{Q}_n^{00}$ such that $\{u, v_{00}\} \cap \mathcal{F}'_{00} = \emptyset$ and a Hamiltonian path γ_1 for $\mathcal{Q}_n^{00} - (\mathcal{F}_{00} \cup \mathcal{F}'_{00})$ from u to v_{01}. In the same way, applying the induction hypothesis for $u_{01}, v_{01} \in \mathcal{V}(\mathcal{Q}_n^{01} - \mathcal{F}_{01})$ we can find a disjoint mirror twin $\mathcal{F}'_{01}$ of $\mathcal{F}_{01}$ in $\mathcal{Q}_n^{01}$ such that $\{u_{01}, v_{01}\} \cap \mathcal{F}'_{01} = \emptyset$ and a Hamiltonian path γ_2 for $\mathcal{Q}_n^{01} - (\mathcal{F}_{01} \cup \mathcal{F}'_{01})$ from u_{01} to v_{01}. Also, applying the induction hypothesis for $u_{11}, v_{11} \in \mathcal{V}(\mathcal{Q}_n^{11} - \mathcal{F}_{11})$ we can find a disjoint mirror twin $\mathcal{F}'_{11}$ of $\mathcal{F}_{11}$ in $\mathcal{Q}_n^{11}$ such that $\{u_{11}, v_{11}\} \cap \mathcal{F}'_{11} = \emptyset$ and a Hamiltonian path γ for $\mathcal{Q}_n^{11} - (\mathcal{F}_{11} \cup \mathcal{F}'_{11})$ from u_{11} to v_{11}.

The length of γ is $2^{n-2} - 2|\mathcal{F}_{11}| - 1$. Since $n \geq 6$ we have that

$$2^{n-2} - 2|\mathcal{F}_{11}| - 1 \geq 2^{n-2} - 2(n-5) - 1 > 2(n-3) \geq 2f.$$

Therefore if we project all vertices from $\mathcal{F}$ onto $\mathcal{Q}_n^{11}$ (parallel to the first two coordinates) there will be at least two consecutive vertices u'_{11} and v'_{11} in γ that are not projections of any vertices from $\mathcal{F}$. Without loss of generality, we can assume that v'_{11} is closer to u_{11} in γ than u'_{11}. Denote by γ_3 the path defined by γ from u_{11} to v'_{11} and by γ_4 the path from u'_{11} to v. Denote also the neighbors of u'_{11} and v'_{11} in $\mathcal{Q}_n^{10}$ by v_{10} and u_{10}.

Since v_{10} and u_{10} do not belong to $\mathcal{F}$ we can apply the induction hypothesis for $u_{10}, v_{10} \in \mathcal{V}(\mathcal{Q}_n^{10} - \mathcal{F}_{10})$ to find a disjoint mirror twin $\mathcal{F}'_{10}$ of $\mathcal{F}_{10}$ in $\mathcal{Q}_n^{10}$, with $\{u_{10}, v_{10}\} \cap \mathcal{F}'_{10} = \emptyset$, and a Hamiltonian path γ_5 for $\mathcal{Q}_n^{10} - (\mathcal{F}_{10} \cup \mathcal{F}'_{10})$ from u_{10} to v_{10}. Then $\mathcal{F}' = \mathcal{F}'_{00} \cup \mathcal{F}'_{01} \cup \mathcal{F}'_{10} \cup \mathcal{F}'_{11}$ is a disjoint mirror twin of $\mathcal{F}$ and the path

$$u \xrightarrow{\gamma_1} v_{00} \to u_{01} \xrightarrow{\gamma_2} v_{01} \to u_{11} \xrightarrow{\gamma_3} v'_{11} \to u_{10} \xrightarrow{\gamma_5} v_{10} \to u'_{11} \xrightarrow{\gamma_4} v$$

is the required path from u to v for $\mathcal{Q}_n - (\mathcal{F} \cup \mathcal{F}')$.

3. Proof of Theorem 1.5

As it is mentioned above, the claim is true for each $f \leq 4$ and each $n \geq f + 2$. We continue the proof by induction on f. Let $f_0 \geq 5$ and assume that our claim is true for each $f < f_0$ and each $n \geq f + 2$. We shall prove our claim for $f = f_0$ and every $n \geq f + 2$. Notice that since $f \geq 5$ we have $n \geq 7$.

Let $f = f_0$ and $n \geq f + 2$. Let also $\mathcal{F}$ be a set of vertices in $\mathcal{Q}_n$ of cardinality f.

Since $f \geq 5$, there must be a coordinate where not all vertices of $\mathcal{F}$ agree. Suppose, without loss of generality, that it is the first coordinate. For $i \in \{0, 1\}$, let $\mathcal{V}_i$ be the subset of $\mathcal{V}(\mathcal{Q}_n)$ consisting of the vertices whose first coordinate is i. Then $1 \leq |\mathcal{V}_i \cap \mathcal{F}| \leq f - 1$ for $i \in \{0, 1\}$, but also, since $f \geq 5$, one of $|\mathcal{V}_0 \cap \mathcal{F}|$ and $|\mathcal{V}_1 \cap \mathcal{F}|$ must be at least three. Suppose, without loss of generality, that $|\mathcal{V}_0 \cap \mathcal{F}| \geq 3$. Then there must be a coordinate, other than the first, where not all vertices of $\mathcal{V}_0 \cap \mathcal{F}$ agree. Without loss of generality, suppose that it is the second coordinate. For $i, j \in \{0, 1\}$, let $\mathcal{V}_{ij}$ be the subset of $\mathcal{V}(\mathcal{Q}_n)$ consisting of the vertices whose first two coordinates are ij, and let $\mathcal{F}_{ij} = \mathcal{V}_{ij} \cap \mathcal{F}$. Since $|\mathcal{V}_0 \cap \mathcal{F}| \geq 3$, either $|\mathcal{F}_{00}|$ or $|\mathcal{F}_{01}|$ must be at least two; suppose without loss of generality that $|\mathcal{F}_{00}| \geq 2$. This means that $\mathcal{V}_{00}$ contains at least two vertices of $\mathcal{F}$ and $\mathcal{V}_{01}$ and $\mathcal{V}_1$ each contain at least one, so we conclude that $|\mathcal{F}_{00}| \leq f - 2$, $|\mathcal{F}_{01}| \leq f - 3$, $|\mathcal{F}_{10}| \leq f - 3$ and $|\mathcal{F}_{11}| \leq f - 3$.

Now, for $i, j \in \{0, 1\}$, let $\mathcal{Q}_n^{ij}$ be the subgraph of $\mathcal{Q}_n$ induced by $\mathcal{V}_{ij}$. Then each $\mathcal{Q}_n^{ij}$ is an $n-2$-dimensional hypercube, and since $|\mathcal{F}_{00}| + 2 \leq (f - 2) + 2 = f \leq n - 2$, we can apply the induction hypothesis for

$\mathcal{Q}_n^{00} - \mathcal{F}_{00}$ to find a disjoint mirror twin $\mathcal{F}'_{00}$ of $\mathcal{F}_{00}$ and a Hamiltonian cycle γ for $\mathcal{Q}_n^{00} - (\mathcal{F}_{00} \cup \mathcal{F}'_{00})$. The length of γ is $2^{n-2} - 2|\mathcal{F}_{00}|$ and since $n \geq 7$ we have

$$2^{n-2} - 2|\mathcal{F}_{00}| \geq 2^{n-2} - 2(n-4) > 2(n-2) \geq 2f.$$

Therefore if we project all vertices from $\mathcal{F}$ onto $\mathcal{Q}_n^{00}$ (parallel to the first two coordinates) there will be at least two consecutive vertices u_{00} and v_{00} in γ that are not projections of any vertices from $\mathcal{F}$. Denote by γ_1 the longer path defined by γ from u_{00} to v_{00}. Denote also the neighbors of u_{00} and v_{00} in $\mathcal{V}_{01}$ and $\mathcal{V}_{10}$, respectively, by v_{01}, u_{01} and v_{10}, u_{10}, and their neighbors in $\mathcal{V}_{11}$ by u_{11}, v_{11}.

Clearly, none of the vertices u_{01}, v_{01}, u_{10}, v_{10}, u_{11}, or v_{11} belong to $\mathcal{F}$. Now, since we know that $|\mathcal{F}_{01}| + 3 \leq (f-3) + 3 = f \leq n-2$, we can apply Theorem 1.6 for $u_{01}, v_{01} \in \mathcal{V}(\mathcal{Q}_n^{01} - \mathcal{F}_{01})$ and to find a disjoint mirror twin $\mathcal{F}'_{01}$ of $\mathcal{F}_{01}$ in $\mathcal{Q}_{01}$ such that $\{u_{01}, v_{01}\} \cap \mathcal{F}'_{01} = \emptyset$ and a Hamiltonian path γ_2 for $\mathcal{Q}_n^{01} - (\mathcal{F}_{01} \cup \mathcal{F}'_{01})$ from u_{01} to v_{01}. In the same way, applying Theorem 1.6 for $u_{11}, v_{11} \in \mathcal{V}(\mathcal{Q}_n^{11} - \mathcal{F}_{11})$ we can find a disjoint mirror twin $\mathcal{F}'_{11}$ of $\mathcal{F}_{11}$ in $\mathcal{Q}_n^{11}$ such that $\{u_{11}, v_{11}\} \cap \mathcal{F}'_{11} = \emptyset$ and a Hamiltonian path γ_3 for $\mathcal{Q}_n^{11} - (\mathcal{F}_{11} \cup \mathcal{F}'_{11})$ from u_{11} to v_{11}. Finally, applying Theorem 1.6 for $u_{10}, v_{10} \in \mathcal{V}(\mathcal{Q}_n^{10} - \mathcal{F}_{10})$ we can find a disjoint mirror twin $\mathcal{F}'_{10}$ of $\mathcal{F}_{10}$ in $\mathcal{Q}_n^{10}$ such that $\{u_{10}, v_{10}\} \cap \mathcal{F}'_{10} = \emptyset$ and a Hamiltonian path γ_4 for $\mathcal{Q}_n^{10} - (\mathcal{F}_{10} \cup \mathcal{F}'_{10})$ from u_{10} to v_{10}. Then $\mathcal{F}' = \mathcal{F}'_{00} \cup \mathcal{F}'_{01} \cup \mathcal{F}'_{10} \cup \mathcal{F}'_{11}$ is a disjoint mirror twin of $\mathcal{F}$ and the cycle

$$u_{00} \xrightarrow{\gamma_1} v_{00} \rightarrow u_{01} \xrightarrow{\gamma_2} v_{01} \rightarrow u_{11} \xrightarrow{\gamma_3} v_{11} \rightarrow u_{10} \xrightarrow{\gamma_4} v_{10} \rightarrow u_{00}$$

is the required Hamiltonian cycle for $\mathcal{Q}_n - (\mathcal{F} \cup \mathcal{F}')$.

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