PATH COVERINGS WITH PRESCRIBED ENDS IN FAULTY HYPERCUBES

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ABSTRACT. We discuss the existence of vertex disjoint path coverings with prescribed ends for the n-dimensional hypercube with or without deleted vertices. Depending on the type of the set of deleted vertices and desired properties of the path coverings we establish the minimal integer m such that for every $n \geq m$ such path coverings exist. Using some of these results, for $k \leq 4$, we prove Locke's conjecture that a hypercube with k deleted vertices of each parity is Hamiltonian if $n \geq k+2$. Some of our lemmas substantially generalize known results of I. Havel and T. Dvořák. At the end of the paper we formulate some conjectures supported by our results.

1. INTRODUCTION

The $n-$dimensional binary hypercube $\mathcal{Q}_n$ is the graph whose vertex set $\mathcal{V}(\mathcal{Q}_n)$ consists of all binary sequences of length n and whose edge set $\mathcal{E}(\mathcal{Q}_n)$ consists of all pairs of binary sequences that differ in exactly one position. In recent years some attention has been given to the problem of finding Hamiltonian cycles or maximal cycles in the $n-$dimensional binary hypercube $\mathcal{Q}_n$ with faulty vertices or with faulty edges.

In [18] Parkhomenko illustrates some techniques of constructing cycles without faulty edges or vertices in low dimensional hypercubes. His methods rely on a classification of Hamiltonian cycles for hypercubes of dimension 4 or less.

Caha and Koubek [8] and Dvořák [10] have addressed the problem of prescribing a set of edges $\mathcal{P}$ through which a Hamiltonian cycle in $\mathcal{Q}_n$ must pass. The best theorem in this direction known to us is the following:

Theorem 1.1. *(Dvořák [10]) Let $\mathcal{P}$ be a set of edges in $\mathcal{Q}_n$ such that each connected component of the subgraph generated by $\mathcal{P}$ is a simple path. If the cardinality of $\mathcal{P}$ is less than or equal to $2n-3$, then there exists a Hamiltonian cycle in $\mathcal{Q}_n$ that passes through each edge in $\mathcal{P}$.*

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Dvořák's proof uses two lemmas about covering the vertices of $\mathcal{Q}_n$ by vertex disjoint paths with prescribed ends. The first one, called Havel's lemma, states that given any two vertices of opposite parity in $\mathcal{Q}_n$, with $n \geq 1$, there exists a Hamiltonian path with these two vertices as endpoints [13, Proposition 2.3]. Dvořák generalizes this lemma as follows:

Lemma 1.2. *(Dvořák [10]) Let $n \geq 2$, a_1, a_2 be two distinct vertices of the same parity, and b_1, b_2 be two distinct vertices of the opposite parity in the hypercube $\mathcal{Q}_n$. Then there exist two vertex-disjoint paths, one joining a_1 to b_1 and the other joining a_2 to b_2, such that each vertex of $\mathcal{Q}_n$ is contained in one of these paths.*

One of the main ingredients in the proof of Dvořák's theorem is the existence of a covering of the vertices of $\mathcal{Q}_n$ by vertex disjoint paths with prescribed end vertices. In this article we address the existence of such path coverings with prescribed end vertices for the hypercube with or without deleted vertices. More specifically, we investigate what is the minimal dimension m of the hypercube $\mathcal{Q}_m$ such that for every $n \geq m$ and every set $\mathcal{F}$ of $M \geq 0$ deleted vertices from $\mathcal{Q}_n$ such that the absolute value of the difference of the numbers of the deleted vertices of the two parities is C, there exists a path covering of $\mathcal{Q}_n - \mathcal{F}$ with N paths whose end vertices are with different parity and O paths whose end vertices are of the same parity, where all of the end vertices of these paths belong to an arbitrary set of non-deleted vertices. The exact meaning of these words can be found in Section 2 where more precise definitions are given including the definition of the symbol $[M, C, N, O]$ that represents the number m mentioned above.

The main results of this paper are contained in the last 4 sections. Section 3 deals with special cases where the numbers M, C, N, and O are small and in many of those cases we use pictorial proofs. In Section 4 we use words to represent paths in the proofs and we study cases of larger numbers of M, C, N, or O. In particular, in that section, we generalize Dvořák's lemma (see Lemma 4.7). Section 5 contains general results that allow us to establish connections between different values of $[M, C, N, O]$. These three sections also contain, for $k \leq 4$, a proof of Locke's conjecture that a hypercube with k deleted vertices of each parity is Hamiltonian if $n \geq k + 2$. In Section 6 we state some conjectures supported by our results and we give some concluding remarks. Appendix A contains a proof of a claim for $n = 4$ that we found difficult to verify by inspection. In a table in Appendix B we summarize many of the results contained in this paper.

2. Some definitions

To simplify the explanations that follow we introduce the following terminology and conventions. *A path covering of a graph* is a set of vertex disjoint paths that cover all the vertices of a given graph. k*−path covering* is a path covering by exactly k paths. Sometimes we call the end vertices of a path *ends* or *terminals.* A vertex of $\mathcal{Q}_n$ is called *even* (*odd*) if it has an even (odd) number of $1's$. A transformation that changes the values of a fixed entry for all the vertices of $\mathcal{Q}_n$ induces an automorphism of the hypercube that sends even vertices to odd vertices and vice versa. Therefore, any statement about $\mathcal{Q}_n$ in terms of even and odd vertices has an equivalent dual statement obtained when the references to even and odd vertices are interchanged. For convenience, we call the vertices of one parity *red* and the vertices of the opposite parity *green* without specifying which are even and which are odd.

A *fault* $\mathcal{F}$ in $\mathcal{Q}_n$ is a set of deleted vertices. The *mass* M of a fault $\mathcal{F}$ is the total number of vertices in the fault. The *charge* C of a fault is the absolute value of the difference between the number of red vertices and the number of green vertices. We say that a fault is *neutral* if its charge is zero. When the endpoints of a path are of the same parity we say that the path is *charged*; otherwise the path is *neutral.* Regarding a pair of vertices we say that the *pair is charged* if the two elements in the pair are of the same parity and that the *pair is neutral* if the two elements are of opposite parity. If the two elements of a charged pair of vertices are red (green) we say that the *pair is red (green).*

Let M be any nonnegative even number and let $\mathcal{A}_M$ be the set of positive integers m with the property that if $n \geq m$ then $\mathcal{Q}_n - \mathcal{F}$ is Hamiltonian for every neutral fault $\mathcal{F}$ of mass M in $\mathcal{Q}_n$. The set $\mathcal{A}_M$ is nonempty (see [17]). We denote by $[M]$ the smallest integer in this set. It is clear that $[0] = 2$ since $\mathcal{Q}_n$ is Hamiltonian if $n \geq 2$, and $[2k] \geq k + 2$ since if k vertices adjacent to a given vertex are removed from $\mathcal{Q}_{k+1}$ then the resulting graph is not Hamiltonian. In Problem 10892 of *The American Mathematical Monthly* [16] S. Locke conjectures that $[2k] = k + 2$ for every nonnegative integer k. A proof of $[2] = 3$ is contained in [17] and a proof of $[4] = 4$ was known to S. Locke (personal communication). To the best of our knowledge Locke's conjecture in its full generality remains unsolved. In Lemmas 3.8, 4.5, and 5.12, we prove that $[2k] = k + 2$ for $k = 2, 3, 4$.

Let $r(\mathcal{F})$ be the number of red vertices and $g(\mathcal{F})$ be the number of green vertices in a fault $\mathcal{F}$ of $\mathcal{Q}_n$. Let also $\mathcal{E}$ be a set of disjoint pairs of vertices of $\mathcal{Q}_n$, $r(\mathcal{E})$ be the number of red pairs in $\mathcal{E}$, and $g(\mathcal{E})$ be the number of green pairs in $\mathcal{E}$. We say that *the set of pairs* $\mathcal{E}$ *is in balance with the fault* $\mathcal{F}$ if all the vertices in the elements of $\mathcal{E}$ are from $\mathcal{Q}_n - \mathcal{F}$ and $r(\mathcal{F}) - g(\mathcal{F}) = g(\mathcal{E}) - r(\mathcal{E})$. Since $\mathcal{Q}_n$ is a bipartite graph with the set of even vertices and the set of odd vertices as partite sets, a necessary condition for a set $\mathcal{E}$ of pairs of vertices to be the set of endpoints of a path covering of $\mathcal{Q}_n - \mathcal{F}$ is that $\mathcal{E}$ to be in balance with $\mathcal{F}$.

Definition 2.1. Let M, C, N, O be nonnegative integers and $\mathcal{F}$ be a fault of mass M and charge C in $\mathcal{Q}_n$. We say that *one can freely prescribe ends for a path covering of* $\mathcal{Q}_n - \mathcal{F}$ *with* N *neutral paths and* O *charged paths* if

(i) there exists at least one set $\mathcal{E}$ of disjoint pairs of vertices that is in balance with $\mathcal{F}$ and contains exactly N neutral pairs and O charged pairs; and
(ii) for every set $\mathcal{E}$ of disjoint pairs of vertices that is in balance with $\mathcal{F}$ and contains exactly N neutral pairs and O charged pairs there exists a path covering of $\mathcal{Q}_n - \mathcal{F}$ such that the set of pairs of end vertices of the paths in the covering coincides with $\mathcal{E}$.

It is easy to see that if in $\mathcal{Q}_n$ there exists a fault $\mathcal{F}$ of mass M and charge C, and a set of pairs of vertices $\mathcal{E}$ that is in balance with $\mathcal{F}$ and contains exactly N neutral pairs and O charged pairs, then $2^n \geq M + C + 2N + 2O$.

Definition 2.2. Let $\mathcal{A}_{M,C,N,O}$ be the set of nonnegative integers m such that

(i) $m \geq \log_2 [M + C + 2N + 2O]$; and
(ii) for every $n \geq m$ and for every fault $\mathcal{F}$ of mass M and charge C in $\mathcal{Q}_n$ one can freely prescribe ends for a path covering of $\mathcal{Q}_n - \mathcal{F}$ with N neutral paths and O charged paths.

We let $[M, C, N, O]$ denote the smallest element in $\mathcal{A}_{M,C,N,O}$ if this set is nonempty.

For example, Havel's lemma quoted above is the statement $[0, 0, 1, 0] = 1$ and Dvořák's lemma is the statement $[0, 0, 2, 0] = 2$.

3. Some cases of small faults or small sets of prescribed end vertices

In the statements below, since only a few vertices are deleted from $\mathcal{Q}_{n+1}$ and we are looking for path coverings with just a few paths, it is convenient to illustrate the proofs by using diagrams. In these diagrams the hypercube $\mathcal{Q}_{n+1}$ is viewed as two copies of the $n-$dimensional hypercube which we call *top plate* and *bottom plate* and we denote by $\mathcal{Q}^{top}_{n+1}$ and $\mathcal{Q}^{bot}_{n+1}$, respectively. The edges connecting the two plates are called *bridges*. We mark on the diagrams only the vertices that are relevant for the proof. To distinguish their colors (parity) we mark the red vertices with stars and leave the green ones unmarked. The prescribed ends of each path are represented by the same geometric figure (triangle, square, etc.) and for different paths we use different figures. The deleted vertices are represented by big circles with a star inside if they are red or a minus inside if they are green. For the proof of a given lemma we usually produce connections on the plates that are guaranteed by previous lemmas or by an induction hypothesis and then we use bridges to connect paths from the top plate to paths from the bottom plate. Sometimes the paths from a plate are cut at certain places and the cut points are connected to the other plate by *bridges*. In such cases we say that *we perform surgery*. The vertices at which we do cuts are represented by tiny circles. The variables $r, r_1, r_2, \ldots$ are reserved to represent red vertices and the variables $g, g_1, g_2, \ldots$ are reserved to represent green vertices.

The following lemma that qualifies $\mathcal{Q}_n$ as a hyper-Hamilton laceable graph was proved by Lewinter and Widulski [15, Corollary 4].

Lemma 3.1. ($[1, 1, 0, 1] = 2$) *Let $n \geq 2$ and d be any vertex in $\mathcal{Q}_n$. Then one can freely prescribe ends for a charged Hamiltonian path of $\mathcal{Q}_n - \{d\}$.*

Corollary 3.2 below is a refinement of Havel's lemma and follows directly from $[0, 0, 2, 0] = 2$ and $[1, 1, 0, 1] = 2$. It also appears as Corollary 3.4 in [10] and therefore is given here without proof.

Corollary 3.2. *Let $n \geq 2$, r and g be a red and a green vertex in $\mathcal{Q}_n$, and e be an edge different from $\{r, g\}$. Then there exists a Hamiltonian path of $\mathcal{Q}_n$ that connects r to g and passes through e.*

The following lemma is a solution to the first part of Problem 10892 proposed by S. Locke in *The American Mathematical Monthly* [16]. For the solution published in *The Monthly* see [17]. We present a different proof.

Lemma 3.3. ($[2] = 3$) *If $n \geq 3$ then $\mathcal{Q}_n - \mathcal{F}$ is Hamiltonian for any neutral fault $\mathcal{F}$ of mass 2.*

Proof. Produce two plates that separate the deleted vertices r and g and assume that the deleted red vertex r is on the top plate. Find two bridges with green vertices on the top plate that do not contain the deleted vertices. Use $[1, 1, 0, 1] = 2$ to produce a Hamiltonian path of $\mathcal{Q}_n^{top} - \{r\}$ that connects the top vertices of the bridges. Use $[1, 1, 0, 1] = 2$ to produce a Hamiltonian path of $\mathcal{Q}_n^{bot} - \{g\}$ that connects the lower vertices of the bridges. The paths produced on the plates connected by the bridges form the desired Hamiltonian cycle in $\mathcal{Q}_n - \mathcal{F}$. □

Lemma 3.4. *Let $n \geq 2$, r be a red vertex and g_1, g_2 be two green vertices in $\mathcal{Q}_n$. Then there are at least $n-1$ Hamiltonian paths of $\mathcal{Q}_n - \{r\}$ that connect g_1 to g_2, all starting with different edges.*

Proof. The proof is by induction. The statement is obvious for $n = 2$. When $n = 3$ there are only two cases to consider: r belongs to the same two dimensional subcube that contains g_1 and g_2 and r does not belong to it. In each one of these cases it is routine to construct the required two paths.

Now let $n \geq 4$. Produce two plates to separate the two green vertices.

Case 1. r and g_1 are on the top plate and g_2 is on the bottom plate.

Let g be any green vertex on the top plate different from g_1 and r_1 be the vertex of Q_n^{bot} that is adjacent to g. By the induction hypothesis there are at least $n-2$ Hamiltonian paths of $\mathcal{Q}_n^{top}-\{r\}$ that connect g_1 to g all starting with different edges from g_1. Extend each of these paths to produce a Hamiltonian path of $\mathcal{Q}_n - \{r\}$ that connects g_1 to g_2 by adding the bridge $\{g, r_1\}$ and then a Hamiltonian path of $\mathcal{Q}_n^{bot}$ that connects r_1 to g_2. The latter path exists since $[0,0,1,0] = 2$. Finally, let r_2 be the vertex of $\mathcal{Q}_n^{bot}$ that is adjacent to g_1. We produce a Hamiltonian path of $\mathcal{Q}_n - \{r\}$ that connects g_1 to g_2 and starts with the bridge $\{g_1, r_2\}$ as follows. Produce a Hamiltonian cycle of $\mathcal{Q}_n^{top} - \{g_1, r\}$. Such cycle exists since $[2] = 3$. Cut this Hamiltonian cycle at two consecutive vertices whose adjacent vertices on $\mathcal{Q}_n^{bot}$ are a green vertex $g_3 \neq g_2$ and a red vertex $r_3 \neq r_2$. Such consecutive vertices exist since the length of the cycle is at least six. Produce a 2−path covering of $\mathcal{Q}_n^{bot}$ with one path connecting r_2 to g_3 and the other connecting r_3 to g_2. Such path covering exists because $[0,0,2,0] = 2$. We obtain the desired Hamiltonian path of $\mathcal{Q}_n - \{r\}$ by adding to the pieces so far produced the bridge $\{g_1, r_2\}$.

Case 2. r and g_2 are on the top plate and g_1 is on the bottom plate.

We can assume that r and g_1 are not adjacent; otherwise, we could separate r, g_1, and g_2 as in Case 1. Let r_1 be the neighbor of g_1 on the top plate, $g_3 \neq g_2$ be any green vertex on the top plate, r_2 be the neighbor of g_3 on the bottom plate, and $g_4 \neq g_1$ be adjacent to r_2 on the bottom plate. According to the induction hypothesis there exist $n-2$ Hamiltonian paths in $\mathcal{Q}_n^{bot} - \{r_2\}$ that connect g_1 to g_4 that all begin with different edges. Similarly, there exist $n-2$ Hamiltonian paths in $\mathcal{Q}_n^{top} - \{r\}$ that connect g_2 to g_3 that all begin with different edges. Let γ be one of these paths. Each Hamiltonian path on the bottom plate could be connected by means of the edge $\{g_4, r_2\}$ and the bridge $\{r_2, g_3\}$ to γ. In that way, we produce $n-2$ Hamiltonian paths of $\mathcal{Q}_n - \{r\}$ connecting g_1 to g_2 and all beginning with different edges.

Now, to produce the $(n-1)$-th Hamiltonian path of $\mathcal{Q}_n - \{r\}$ that connects g_1 to g_2 and begins with a different edge we proceed as follows. Produce a Hamiltonian path of $\mathcal{Q}_n^{top}$ that connects r_1 to g_2. Cut this path just before and right after r and produce two bridges. Let their ends on the bottom plate be r_3 and r_4. Then there exists a Hamiltonian path for $\mathcal{Q}_n^{bot} - \{g_1\}$ that connects r_3 to r_4 ($[1,1,0,1] = 2$). Then the desired Hamiltonian path of $\mathcal{Q}_n - \{r\}$ that connects g_1 to g_2 is obtained by connecting the paths constructed on the plates by means of the bridges after removing the edges incident to r from the path on the top plate and attaching the edge $\{g_1, r_1\}$ to the resulting path. □

Let a be a vertex in $\mathcal{Q}_n$. There is a unique vertex $\bar{a}$ in $\mathcal{Q}_n$ at distance n from a. The coordinates of $\bar{a}$ are the negation of the corresponding coordinates of a.

Let $\{r, g\}$ be a pair of a red and a green vertex in $\mathcal{Q}_3$. We define the set of pairs of vertices $\mathcal{B}_{\{r,g\}}$ in the following way: if $r = \bar{g}$ then $\{r', g'\} \in \mathcal{B}_{\{r,g\}}$ if and only if $\{r', g'\} \neq \{r, g\}$ and $r' = \bar{g'}$; if $r \neq \bar{g}$ then $\mathcal{B}_{\{r,g\}} = \{\{\bar{r}, \bar{g}\}\}$.

Lemma 3.5. *Let r, g be a red and a green vertex in $\mathcal{Q}_3$, and let r_1, g_1 be a red and a green vertex in $\mathcal{Q}_3 - \{r, g\}$. Then*

(1) *If* $\{r_1, g_1\} \notin \mathcal{B}_{\{r,g\}}$ *then there exists a Hamiltonian path of* $\mathcal{Q}_3 - \{r, g\}$ *that connects* r_1 *to* g_1.
(2) *If* $\{r_1, g_1\} \in \mathcal{B}_{\{r,g\}}$ *then there does not exist a Hamiltonian path of* $\mathcal{Q}_3 - \{r, g\}$ *that connects* r_1 *to* g_1.
(3) *If* $\{r_1, g_1\} \in \mathcal{B}_{\{r,g\}}$ *then there exist two distinct* $2-$*path coverings of* $\mathcal{Q}_3 - \{r, g\}$, *with four distinct end points, with one path starting at* r_1, *the other starting at* g_1, *and both paths of length two.*
(4) *There exist two distinct* $3-$*path coverings of* $\mathcal{Q}_3 - \{r, g\}$ *with paths of length one.*

Proof. By inspection. □

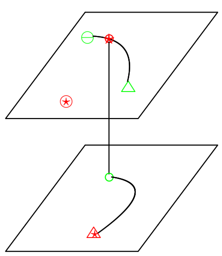
Lemma 3.6. ($[2, 0, 1, 0] = 4$) *Let* $n \geq 2$ *and* r, r_1, g, g_1 *be two red and two green vertices in* $\mathcal{Q}_n$. *If* $n = 2$ *or* $n \geq 4$ *then there exists a Hamiltonian path for* $\mathcal{Q}_n - \{r_1, g_1\}$ *connecting* r *to* g. *If* $n = 3$ *the same conclusion follows provided* $\{r, g\} \notin \mathcal{B}_{\{r_1,g_1\}}$.

Proof. The statement is obvious for $n = 2$ and for $n = 3$ the claim is contained in Lemma 3.5(1). Also, Lemma 3.5(2) shows that $[2, 0, 1, 0] \geq 4$.

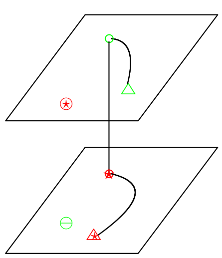
Now, let $n \geq 4$. Produce two plates to separate r from r_1 and assume that r_1 is on the top plate. Then g and g_1 can be distributed in four different ways:

(1) both are on the top plate;
(2) g is on the top plate and g_1 is on the bottom plate;
(3) g_1 is on the top plate and g is on the bottom plate; and
(4) both are on the bottom plate.

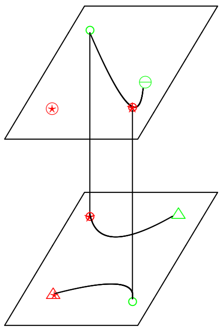
The following diagrams show how to handle these cases.



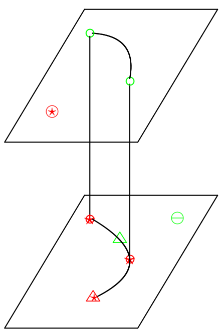
(1) Use $[1, 1, 0, 1] = 2$ to produce a path covering of the top plate connecting the green terminal g to the deleted green vertex g_1 avoiding the deleted red vertex r_1. Cut this path just before the deleted green vertex and produce a bridge from the cut vertex. Use $[0, 0, 1, 0] = 1$ to produce a Hamiltonian path of the bottom plate that connects the lower vertex of the bridge to the red terminal r.



(2) Find a bridge with green vertex on the top different from g and red vertex on the bottom different from r. Use $[1, 1, 0, 1] = 2$ to connect the green terminal to the bridge avoiding the red deleted vertex. Use $[1, 1, 0, 1] = 2$ to produce a Hamiltonian path of the bottom plate that connects the lower vertex of the bridge to the red terminal avoiding the deleted green vertex.



(3) Find a bridge with green vertex on the top different from g_1 and red vertex on the bottom different from r. Use $[1, 1, 0, 1] = 2$ and Lemma 3.4 to connect the upper vertex of the bridge to the deleted green vertex avoiding the deleted red vertex and making sure that the vertex immediately next to the deleted green vertex along the path is not adjacent to the green terminal on the bottom plate. Cut the path just before the deleted green vertex and produce a bridge from the cut vertex. Use $[0, 0, 2, 0] = 2$ to produce a $2-$path covering of the bottom plate that connects the lower vertices of the bridges to the appropriate terminals.



(4) Find a bridge with a red vertex on the bottom plate different from r. Use $[1, 1, 0, 1] = 2$ to connect the red terminal on the bottom plate to the lower vertex of the bridge avoiding the green deleted vertex. This path must pass through the green terminal. Cut the path just before the green terminal and produce another bridge at the cut vertex. On the top plate use $[1, 1, 0, 1] = 2$ to connect the upper vertices of the bridges avoiding the red deleted vertex. □

Corollary 3.7. *Let $n \geq 4$ and $\mathcal{F}$ be any neutral fault of mass 2 in $\mathcal{Q}_n$. Then for any edge e in $\mathcal{Q}_n - \mathcal{F}$ there exists a Hamiltonian cycle of $\mathcal{Q}_n - \mathcal{F}$ that contains e.*

Lemma 3.8. ($[4] = 4$) *Let $n \geq 4$ and $\mathcal{F}$ be any neutral fault of mass 4 in $\mathcal{Q}_n$. Then $\mathcal{Q}_n - \mathcal{F}$ is Hamiltonian. The claim is not true for $n = 3$.*

Proof. Since $[2k] \geq k + 2$ for each integer $k \geq 0$, we have $[4] \geq 4$.

Let $n \geq 4$, r_1, r_2 be the two red, and g_1, g_2 be the two green vertices in $\mathcal{F}$. Split $\mathcal{Q}_n$ into two plates with r_1 on the top plate and r_2 on the bottom plate. There are two essentially different cases that depend on the distribution of the green deleted vertices between the plates.

Case 1. The two deleted green vertices are on the top plate.

Use $[1, 1, 0, 1] = 2$ to produce a path on the top plate that connects the two deleted green vertices and visits all the vertices of the top plate except the deleted red vertex. From the vertices immediately next to the deleted green vertices along the constructed path, produce bridges to connect to the bottom plate. Use $[1, 1, 0, 1] = 2$ to connect the lower vertices of these bridges by a path on the bottom plate that visits all the vertices of the bottom plate except the deleted red vertex. To produce the desired Hamiltonian cycle in $\mathcal{Q}_n - \mathcal{F}$ remove from the path constructed on the top plate the edges connecting to the deleted green vertices and attach to the resulting path, by means of the bridges, the path constructed on the bottom plate.

Case 2. g_1 is on the top plate and g_2 is on the bottom plate.

We produce a Hamiltonian cycle of $\mathcal{Q}_n^{top} - \{r_1, g_1\}$ using $[2] = 3$. Along this cycle find two consecutive vertices r_3, g_3 with adjacent vertices on the bottom plate g_4 and r_4, respectively, with $g_4 \neq g_2$ and $r_4 \neq r_2$, and such that g_4 is adjacent to r_2. This last requirement is important for $n = 4$ but irrelevant for higher dimensions. It guarantees that $\{r_4, g_4\} \notin \mathcal{B}_{\{r_2,g_2\}}$ when the bottom plate is isomorphic to $\mathcal{Q}_3$. (To see that such vertices r_3 and g_3 exist just take g_4 to be a neighbor of r_2 in $\mathcal{Q}_n^{bot} - \{g_2\}$ which is not a neighbor of r_1 (since $n \geq 4$ such a neighbor exists). Then denote by r_3 the neighbor of g_4 in $\mathcal{Q}_n^{top}$. Clearly r_3 will be different from r_1 and will belong to the Hamiltonian cycle on the top. Now take g_3 to be a neighbor of r_3 in that cycle which is not a neighbor of r_2.) Then using $[2, 0, 1, 0] = 4$ we can produce a Hamiltonian path of $\mathcal{Q}_n^{bot} - \{r_2, g_2\}$ that connects r_4 to g_4. The desired Hamiltonian cycle of $\mathcal{Q}_n - \mathcal{F}$ is formed by connecting the path on the bottom plate to the cycle on the top plate by mean of the bridges $\{r_3, g_4\}, \{r_4, g_3\}$ and, of course, removing the edge $\{r_3, g_3\}$. □

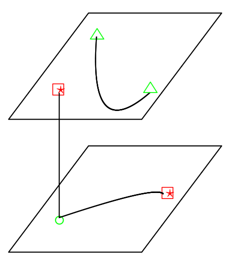
For the sake of brevity, from now on, we adopt the following conventions for the proofs using diagrams. The paths drawn on each plate are assumed to form path coverings of that plate so we indicate in the diagram just what vertices are connected by these paths. From the diagram it will be clear which vertices are avoided by the path covering. A sentence such as "we find a bridge with green at the top" means that we select a green vertex on the top plate such that neither it nor its adjacent vertex on the bottom plate is a terminal or a deleted vertex, and we produce the bridge between these two vertices. A sentence such as "we choose two adjacent bridges along this path to do surgery" means that 1) we select two consecutive vertices along the mentioned path such that neither them nor their adjacent vertices on the other plate are terminals or deleted vertices; 2) we produce bridges from the selected vertices to the other plate; and 3) we remove the edge that connects the selected vertices. At the end of each construction, when we produce the final path covering, all the edges of the original path covering that were connected to deleted vertices, if such edges exist, must be cut out. The desired path covering is formed by the paths that connect figures of the same color and shape to each other. These paths should be clear to the reader from the diagrams.

The following lemma was independently obtained by Caha and Koubek [9, Corollary 10]. However their proof is too involved. We provide here a simpler and direct proof.

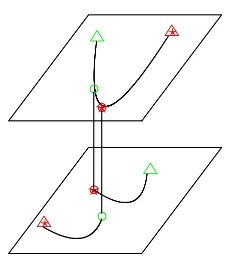
Lemma 3.9. ($[0, 0, 0, 2] = 4$) *Let $n \geq 3$ and r, r_1, g, g_1 be two red and two green vertices in $\mathcal{Q}_n$. If $n \geq 4$ then there exists a $2-$path covering of $\mathcal{Q}_n$ with one path connecting r to r_1 and the other connecting g to g_1. If $n = 3$ the same conclusion holds provided that r and r_1 are contained in a* *two dimensional subcube α of $\mathcal{Q}_3$ and exactly one of the vertices g or g_1 is contained in α.*

Proof. The claim is straightforward for $n = 3$. Also, one can directly verify that if r and r_1 are contained in a two dimensional subcube α of $\mathcal{Q}_3$ and none or both of the vertices g and g_1 are contained in α then there does not exist a $2-$path covering of $\mathcal{Q}_3$ with one path connecting r to r_1 and the other connecting g to g_1. Therefore $[0, 0, 0, 2] \geq 4$.

Let $n \geq 4$. Split $\mathcal{Q}_n$ into two plates that separate the two red terminals. We can assume that $r \in \mathcal{Q}_n^{top}$ and $r_1 \in \mathcal{Q}_n^{bot}$. There are two essentially different cases that depend on the distribution of the green terminals between the plates: (1) the two green terminals are on the top plate; and (2) g is on the top plate and g_1 is on the bottom plate. These cases can be handled as explained in the following diagrams.



(1) Use $[1, 1, 0, 1] = 2$ to find a Hamiltonian path of $\mathcal{Q}_n^{top} - \{r\}$ that connects g to g_1. Connect the top red terminal r to the bottom plate by a bridge. Use $[0, 0, 1, 0] = 1$ to find a Hamiltonian path of the bottom plate that connects the lower vertex of the bridge to the red terminal r_1 on the bottom plate.



(2) Use $[0, 0, 1, 0] = 1$ to produce a Hamiltonian path of the top plate that connects the two terminals. While traversing the path starting from the green terminal, find an edge whose first vertex is green and such that the adjacent vertices on the bottom are not terminals. Produce bridges from the vertices of this edge. Use $[0, 0, 2, 0] = 2$ to produce a $2-$path covering of the bottom plate that connects the lower vertices of the bridges to the appropriate terminals. □

The following lemma is a refinement of Lemma 3.9. It shows that one can choose which one of the two pairs of terminals to be connected by the longer path.

Lemma 3.10. *Let $n \geq 3$, r, r_1 be two distinct red vertices and g, g_1 be two distinct green vertices in $\mathcal{Q}_n$. If $n = 3$ we also require that if r and r_1 are contained in a two dimensional subcube α of $\mathcal{Q}_3$, then exactly one of the vertices g or g_1 is contained in α. Then there exists a $2-$path covering of $\mathcal{Q}_n$ with the first path of length at least 2^{n-1} connecting r to r_1 and the second path connecting g to g_1.*

Proof. If $n = 3$ then our claim can be verified directly.

For $n \geq 4$ we produce two plates as in the proof of Lemma 3.9 and consider the same two cases. The proof of case (1) does not need to be modified. For case (2) we assume without loss of generality that r, g are on the top plate and r_1, g_1 are on the bottom plate. There are three subcases to consider.

Subcase 2(a). g is not adjacent to r_1.

Let r_2 be any red vertex on the top plate that is adjacent to vertex g_2 of the bottom plate different from g_1. Use $[1, 1, 0, 1] = 2$ to produce a Hamiltonian path of $\mathcal{Q}_n^{top} - \{g\}$ that connects r to r_2. Let r_3 be the vertex of the bottom plate that is adjacent to g. Use $[0, 0, 2, 0] = 2$ to produce a $2-$path covering of the bottom plate that connects r_3 to g_1 and g_2 to r_1. The desired $2-$path covering of $\mathcal{Q}_n$ is obtained by connecting the path produced on the plates by means of the bridges $\{r_2, g_2\}$ and $\{g, r_3\}$.

Subcase 2(b). r is not adjacent to g_1.

Let r_2 be the vertex of the top plate that is adjacent to g_1. Let g_2 be any green vertex on the top plate different from g and adjacent to a vertex $r_3 \neq r_1$ of the bottom plate. Use $[0, 0, 2, 0] = 2$ to produce a $2-$path covering of $\mathcal{Q}_n^{top}$ that connects g to r_2 and r to g_2. Use $[1, 1, 0, 1] = 2$ to produce a Hamiltonian path of $\mathcal{Q}_n^{bot} - \{g_1\}$ that connects r_3 to r_1. The desired $2-$path covering of $\mathcal{Q}_n$ is obtained by attaching to the paths constructed on the plates the bridges $\{r_2, g_1\}$ and $\{g_2, r_3\}$.

Subcase 2(c). r is adjacent to g_1 and r_1 is adjacent to g.

The care in choice of vertices below is important for dimension $n = 4$ but can be relaxed for $n \geq 5$.

Let r_2 be any vertex of the bottom plate that is adjacent to g_1, different from r_1, and let g_2 be the vertex on the top plate that is adjacent to r_2. On the top plate we can find a vertex r_3 whose adjacent vertex g_3 on the bottom plate satisfies the following conditions: 1) g_3 is adjacent to r_2; 2) $g_3 \neq g_1$; and, in the case $n = 4$, we also require 3) the two-dimensional subcube that contains r and r_3 contains exactly one of the vertices g or g_2. Conditions 1) and 2) guarantee the existence of a Hamiltonian path of $\mathcal{Q}_n^{bot} - \{r_2, g_1\}$ that connects g_3 to r_1. Condition 3) guarantees the existence of a $2-$path covering of $\mathcal{Q}_n^{top}$ that connects g to g_2 and r to r_3. The desired $2-$path covering of $\mathcal{Q}_n$ is obtained by attaching to the paths constructed on the plates the bridges $\{r_2, g_2\}, \{g_3, r_3\}$ and the edge $\{r_2, g_1\}$. □

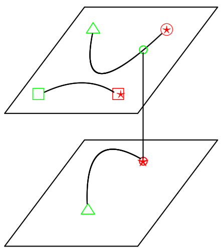
Lemma 3.11. ($[1, 1, 1, 1] = 4$) *Let $n \geq 4$, r be a deleted red vertex in $\mathcal{Q}_n$, and r_1, g, g_1, g_2 be one red and three distinct green vertices in $\mathcal{Q}_n - \{r\}$. Then there exists a $2-$path covering of $\mathcal{Q}_n - \{r\}$ with one path connecting r_1 to g and the other connecting g_1 to g_2. The claim is not true for $n = 3$.*

Proof. The following counterexample shows that $[1, 1, 1, 1] > 3$: $n = 3$, $r = (1, 0, 1)$, $r_1 = (1, 1, 0)$, $g = (1, 1, 1)$, $g_1 = (0, 1, 0)$, $g_2 = (0, 0, 1)$.

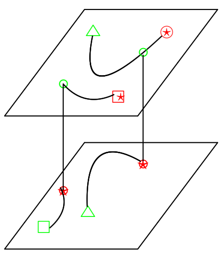
Now let $n \geq 4$. Produce two plates to separate the two green terminals g_1 and g_2 of the charged path and assume that the deleted red r and g_1 are on the top plate. The terminals of the neutral path r_1 and g could be distributed in four possible ways:

(1) both are on the top plate;
(2) the red is on the top plate and the green is on bottom plate;
(3) the green is on the top plate and the red is on the bottom plate;
(4) both are on the bottom plate.

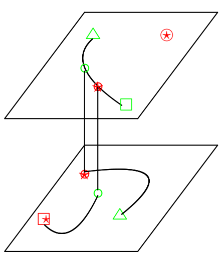
The four cases can be approached as explained in the following diagrams.



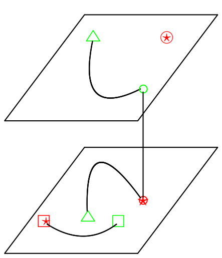
(1) Use $[0,0,2,0] = 2$ to produce a $2-$path covering of the top plate that connects the two terminals of the neutral path to each other, and the green terminal of the charged path to the deleted red. Cut the last path just before the deleted red and produce a bridge. Use $[0,0,1,0] = 1$ to find a Hamiltonian path for the bottom plate connecting the lower vertex of the bridge to the green terminal on the bottom plate.



(2) Find a bridge with green on the top. Use $[0,0,2,0] = 2$ to produce a $2-$path covering of the top plate that connects the red terminal of the neutral path to the upper vertex of the bridge and the green terminal of the charged path to the deleted red vertex. Cut the second path just before the deleted red vertex and produce a second bridge there. Use $[0,0,2,0] = 2$ to produce a $2-$path covering of the bottom plate that connects the lower vertices of the bridges to the appropriate terminals.



(3) Use $[1,1,0,1] = 2$ to produce a Hamiltonian path of $\mathcal{Q}_n^{top} - \{r\}$ that connects g to g_1. Traversing this path from g to g_1 find two consecutive vertices that are not neighbors to the green and red terminals on the bottom plate and such that the first vertex is green. Such pair of consecutive vertices exist since the length of the path is at least six, hence there are at least three such pairs on the top and only two vertices to avoid on the bottom. Produce bridges from these vertices. Use $[0,0,2,0] = 2$ to produce a $2-$path covering of the bottom plate that connects the lower vertices of the bridges to the appropriate terminals.



(4) Find a bridge with green on the top. Use $[1,1,0,1] = 2$ to find a Hamiltonian path of the top plate connecting the green terminal of the charged path to the bridge avoiding the deleted red vertex. Use $[0,0,2,0] = 2$ to find a $2-$path covering of the bottom plate connecting the lower vertex of the bridge to the green terminal of the charged path and the two terminals of the neutral path. $\square$

Lemma 3.12. *Let $n \geq 4$, r_1 and r_2 be two distinct red vertices in $\mathcal{Q}_n$ and g be a green vertex that is deleted from $\mathcal{Q}_n$. Assume further that $e = \{a, b\}$ is any edge in $\mathcal{Q}_n - \{g\}$. Then there exists a Hamiltonian path in $\mathcal{Q}_n - \{g\}$ that connects r_1 to r_2 and passes through the edge e. In the case when $\{a, b\} \cap \{r_1, r_2\} = \emptyset$ we can find an oriented Hamiltonian path in $\mathcal{Q}_n - \{g\}$ connecting r_1 to r_2 such that the path visits the vertex a first.*

Proof. If the prescribed edge e is not incident to any of the prescribed end vertices r_1, r_2, use $[1, 1, 1, 1] = 4$ to connect r_1 to a and r_2 to b. The desired (oriented) Hamiltonian path in $\mathcal{Q}_n - \{g\}$ is obtained by connecting these two paths to each other through the edge e.

Let the prescribed edge be incident to one of the prescribed end vertices. We can assume without loss of generality that $a = r_1$. Then use $[2, 0, 1, 0] = 4$ to produce a Hamiltonian path in $\mathcal{Q}_n - \{r_1, g\}$ that connects r_2 to b. Then attach the edge e to this path to obtain the desired Hamiltonian path in $\mathcal{Q}_n - \{g\}$. $\square$

Lemma 3.13. ($[3, 1, 0, 1] = 4$) *Let $n \geq 4$ and g, r and r_1 be one green and two distinct red vertices in $\mathcal{Q}_n$. Let also g_1 and g_2 be two distinct green terminals in $\mathcal{Q}_n - \{g, r, r_1\}$. Then there exists a Hamiltonian path for $\mathcal{Q}_n - \{g, r, r_1\}$ connecting g_1 to g_2. The claim is not true for $n = 3$.*

Proof. The following counterexample shows that $[3, 1, 0, 1] > 3$: $n = 3$, $r = (1, 0, 1)$, $r_1 = (1, 1, 0)$, $g = (1, 1, 1)$, $g_1 = (0, 1, 0)$, $g_2 = (0, 0, 1)$.

Now, let $n \geq 4$. There exist two plates that separate the deleted red vertices r and r_1 and we assume that the top plate is the one that contains the deleted green vertex g. We consider the three essentially different cases that depend on the distribution of the green terminals g_1 and g_2 on the plates.

Case 1. The two green terminals are on the top plate.

Use $[1, 1, 0, 1] = 2$ to produce a path that visits all the vertices of the top plate except the red deleted vertex and starts at one green terminal and ends at the deleted green vertex. This path must pass through the second green terminal. Cut this path at the vertex immediately preceding the second green terminal and at the vertex immediately preceding the deleted green vertex along the path. From the cut vertices produce two bridges. The lower vertices of these bridges are green. Connect them by a path on the bottom plate that visits all the vertices except the deleted red vertex. This finishes the construction of the desired path for this case.

Case 2. One green terminal is on the top plate and the other one is on the bottom plate.

Use $[1, 1, 0, 1] = 2$ to produce a path on the top plate that visits all the vertices except the deleted red vertex and that starts at the green terminal and ends at the deleted green vertex. By Lemma 3.4 this path can be chosen in such a way that the vertex just before the deleted green is not adjacent to the green terminal on the bottom. Cut the path just before the deleted green and produce a bridge from the cut vertex. Use $[1, 1, 0, 1] = 2$ to produce a path on the bottom plate that connects the lower vertex of the bridge to the green terminal and that visits all the vertices of the bottom plate except the red deleted vertex.

Case 3. The two green terminals are on the bottom plate.

Use $[2] = 3$ to produce a cycle on the top plate that visits all the vertices except the deleted ones.

If $n = 4$, use $[1, 1, 0, 1] = 2$ to produce a path on the bottom plate that visits all the vertices except the deleted red vertex and has the two green terminals as end vertices. At least one non-terminal vertex u of this path is adjacent to a vertex v in the cycle on the top plate. Since the degree of each of these vertices relative to its plate is three, one of the neighbors of u in the bottom path must be adjacent to one of the neighbors of v in the cycle produced on the top plate. In other words, there exist two parallel bridges such that the edges connecting their ends on the bottom and on the top plate belong to the path on the bottom plate and to the cycle on the top plate, respectively. Use these bridges to do surgery to connect the bottom path to the cycle on the top plate by means of the bridges. This finishes the construction of the desired path for this case when $n = 4$.

If $n \geq 5$ then the plates are of dimension greater than three. Thus, there exist two consecutive vertices along the cycle constructed on the top plate such that their adjacent vertices on the bottom plate are neither deleted vertices nor terminal vertices. Select two such vertices and cut the cycle there and produce bridges to the bottom plate. Then use Lemma 3.12 to produce a path on the bottom plate that 1) starts at one green terminal and ends at the other green terminal; 2) visits all the vertices of the bottom plate except the deleted red vertex; and 3) passes through the edge incident to the lower vertices of the two bridges. Finally, do surgery to connect the path on the bottom plate to the cycle on the top plate through the bridges. The result is the desired path. This finishes the construction of the desired path for this case when $n \geq 5$. □

Lemma 3.14. *Let $n \geq 4$ and g and r be a green and a red vertex in $\mathcal{Q}_n$. Let also g_1 and g_2 be two distinct green vertices in $\mathcal{Q}_n - \{g, r\}$. Then there exists a Hamiltonian cycle for $\mathcal{Q}_n - \{g, r\}$ such that the shortest distance between g_1 and g_2 along that cycle is at least four.*

Proof. Split $\mathcal{Q}_n$ into two plates such that g_1 is on the top plate and g_2 is on the bottom plate. There are two cases to consider.

Case 1. r and g are on the top plate.

Use $[2] = 3$ to find a Hamiltonian cycle for $\mathcal{Q}_n^{top} - \{r, g\}$. Choose an edge (g_3, r_3) from this cycle such that $g_1 \neq g_3$ and r_3 is not adjacent to g_2. Cut the cycle at that edge and connect the resulting path with bridges to the bottom plate. Use $[0, 0, 1, 0] = 1$ to find a Hamiltonian path for the bottom plate that connects the bottom vertices of the two bridges. The resulting Hamiltonian cycle of $\mathcal{Q}_n - \{g, r\}$ has the required property.

Case 2. r is on the top plate, g is on the bottom plate.

Find two bridges with green vertices on the top plate that avoid g_1. Use $[1, 1, 0, 1] = 2$ to find Hamiltonian paths for $\mathcal{Q}_n^{top} - \{r\}$ and $\mathcal{Q}_n^{bot} - \{g\}$, respectively, that connect the end vertices of the bridges. The resulting Hamiltonian cycle of $\mathcal{Q}_n - \{g, r\}$ has the required property. □

4. Larger faults and sets of prescribed ends

In this section we identify the hypercube $\mathcal{Q}_n$ with the group $\mathbf{Z}_2^n$. We view $\mathcal{Q}_n$ as a Cayley graph with the standard system of generators $\mathbf{S} = \{e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)\}$. An oriented edge in $\mathcal{Q}_n$ is represented by (a, x), where a is the starting vertex and x is an element from the system of generators $\mathbf{S}$. A path is represented by (a, ω), where a is the initial vertex and ω is a word with letters from $\mathbf{S}$. If $\omega = x_1, x_2, \ldots, x_k$ then the path (a, ω) is the path $a, ax_1, ax_1x_2, \ldots, ax_1x_2 \cdots x_n$. The algebraic content of a word ω is the element of $\mathbf{Z}_2^n$ that is obtained by multiplying all the letters of ω. A path (a, ω) is simple if no subword of ω is algebraically equivalent to the identity $(0, 0, \ldots, 0)$. A path (a, ω) is a cycle if ω is algebraically equivalent to the identity but no proper subword of ω is algebraically equivalent to the identity.

We shall use the following notation: ω^R means the reverse word of ω; ω' denotes the word obtained after the last letter is deleted from ω; ω^* is the word obtained after the first letter is deleted from ω; $\varphi(\omega)$ is the first letter of ω, and $\lambda(\omega)$ is the last letter of ω. The letter v shall be reserved for steps connecting two plates. The letters $x, y, \ldots$ shall be reserved to represent steps along the plates.

The following lemma can be proved by inspection.

Lemma 4.1. *Let r, r_1, r_2 be three distinct red vertices and g, g_1, g_2 be three distinct green vertices in $\mathcal{Q}_3$. Then there exist two oriented paths γ_1, γ_2 such that*

(i) *γ_1 is Hamiltonian in $\mathcal{Q}_3 - \{g\}$ and connects r_1 to r_2;*
(ii) *γ_2 is Hamiltonian in $\mathcal{Q}_3 - \{r\}$ and connects g_1 to g_2; and*
(iii) *γ_1 and γ_2 share an edge that is traversed in the same direction in both paths.*

The following lemma is a generalization of Lemma 4.1.

Lemma 4.2. *Let $n \geq 4$ and r_1, r_2, g_1, g_2, g_3, g_4 be two distinct red and four distinct green vertices in $\mathcal{Q}_n$ such that $r_1, g_1, g_2 \in \mathcal{Q}_n^{top}$ and $r_2, g_3, g_4 \in \mathcal{Q}_n^{bot}$. Then there exist two oriented paths γ_1, γ_2 such that*

(i) *γ_1 is Hamiltonian in $\mathcal{Q}_n^{top} - \{r_1\}$ and connects g_1 to g_2;*

(ii) γ_2 *is Hamiltonian in* $\mathcal{Q}_n^{bot} - \{r_2\}$ *and connects* g_3 *to* g_4*; and*

(iii) *there exist an edge* $(a, ax) \in \gamma_1$ *such that* $(av, avx) \in \gamma_2$ *and both edges are traversed in the same direction in both paths.*

Proof. The proof is by induction. If $n = 4$ then the claim is contained in Lemma 4.1. If $n > 4$ then choose an edge $(a, ax) \in \mathcal{Q}_n^{top}$ such that none of the given vertices r_1, r_2, g_1, g_2, g_3, g_4 is incident to (a, ax) or (av, avx) and apply Lemma 3.12 to construct γ_1 and γ_2 in the desired way. □

Lemma 4.3. ($[2, 2, 0, 2] = 4$) *Let* $n \geq 4$, $\mathcal{F} = \{r_1, r_2\}$ *be a fault with two distinct red vertices and* g_1, g_2, g_3, g_4 *be four distinct green vertices in* $\mathcal{Q}_n$*. Then there exists a* $2-$*path covering of* $\mathcal{Q}_n - \mathcal{F}$ *with one path connecting* g_1 *to* g_2 *and the other connecting* g_3 *to* g_4*. The claim is not true for* $n = 3$.

Proof. The following counterexample shows that $[2, 2, 0, 2] > 3$: $n = 3$, $r_1 = (1, 1, 0)$, $r_2 = (1, 0, 1)$, $g_1 = (0, 1, 0)$, $g_2 = (0, 0, 1)$, $g_3 = (1, 0, 0)$, $g_4 = (1, 1, 1)$.

Now let $n \geq 4$. Split the hypercube in such a way that r_1 is on the top plate and r_2 is on the bottom plate. Then consider four cases that depend on the distribution of the green terminals on the plates.

Case 1. All green terminals g_1, g_2, g_3, g_4 are on the top plate.

Use $[1, 1, 0, 1] = 2$ to find a Hamiltonian path (g_1, ω) of $\mathcal{Q}_n^{top} - \{r_1\}$ that connects g_1 to g_2. Let $\omega = \xi\eta\theta$ with $g_1\xi = g_3, g_3\eta = g_4$, and $g_4\theta = g_2$, where g_3, g_4 are renumbered, if necessary.

Use $[1, 1, 0, 1] = 2$ to find a Hamiltonian path $(g_1\xi' v, \mu)$ of $\mathcal{Q}_n^{bot} - \{r_2\}$ that connects $g_1\xi' v$ to $g_1\xi\eta\varphi(\theta)v$. Then the desired $2-$path covering of $\mathcal{Q}_n - \mathcal{F}$ is $(g_1, \xi' v\mu v\theta^*)$, (g_3, η).

Case 2. g_1, g_2, g_3 are on the top and g_4 is on the bottom plate.

Use $[1, 1, 0, 1] = 2$ to find a Hamiltonian path (g_1, ω) of $\mathcal{Q}_n^{top} - \{r_1\}$ that connects g_1 to g_3. Let $\omega = \xi\eta$, where $g_1\xi = g_2$ and $g_2\eta = g_3$.

Subcase 2(a). $g_2\varphi(\eta)v \neq g_4$.

On the bottom plate use again $[1, 1, 0, 1] = 2$ to find a Hamiltonian path $(g_2\varphi(\eta)v, \mu)$ of $\mathcal{Q}_n^{bot} - \{r_2\}$ that connects $g_2\varphi(\eta)v$ to g_4. Then the desired $2-$path covering of $\mathcal{Q}_n - \mathcal{F}$ is (g_1, ξ), $(g_3, (\eta^R)' v\mu)$.

Subcase 2(b). $g_2\varphi(\eta)v = g_4$.

Either g_1 or g_2 is not adjacent to r_2. Without loss of generality assume that it is g_1. If $n \geq 5$, use $[2, 0, 1, 0] = 4$ to find a Hamiltonian path $(g_1 v, \mu)$ of $\mathcal{Q}_n^{bot} - \{r_2, g_4\}$ that connects $g_1 v$ to $g_1\varphi(\xi)v$. Then the desired $2-$path covering of $\mathcal{Q}_n - \mathcal{F}$ is $(g_1, v\mu v\xi^*)$, $(g_3, (\eta^R)' v)$.

The same argument works for $n = 4$ whenever $\{g_1 v, g_1\varphi(\xi)v\} \notin \mathcal{B}_{\{g_4, r_2\}}$ (Lemma 3.6). If $\{g_1 v, g_1\varphi(\xi)v\} \in \mathcal{B}_{\{g_4, r_2\}}$ then the distance from $g_1\varphi(\xi)v$ to r_2 is three and therefore $g_1\varphi(\xi^*)v \neq r_2$ and $\{g_1\varphi(\xi)v, g_1\varphi(\xi^*)v\} \notin \mathcal{B}_{\{g_4, r_2\}}$. Then use Lemma 3.6 to find a Hamiltonian path $(g_1\varphi(\xi)v, \mu)$ of $\mathcal{Q}_n^{bot} - \{r_2, g_4\}$ that connects $g_1\varphi(\xi)v$ to $g_1\varphi(\xi^*)v$. The desired $2-$path covering of $\mathcal{Q}_n - \mathcal{F}$ is $(g_1, \varphi(\xi)v\mu v\xi^{**})$, $(g_3, (\eta^R)' v)$.

Case 3. g_1, g_2 are on the top and g_3, g_4 are on the bottom plate.

Use $[1, 1, 0, 1] = 2$ to find a Hamiltonian path (g_1, ω) of $\mathcal{Q}_n^{top} - \{r_1\}$ that connects g_1 to g_2 and use again $[1, 1, 0, 1] = 2$ to find a Hamiltonian path (g_3, μ) of $\mathcal{Q}_n^{bot} - \{r_2\}$ that connects g_3 to g_4. Then the desired 2−path covering of $\mathcal{Q}_n - \mathcal{F}$ is (g_1, ω), (g_3, μ).

Case 4. g_1, g_3 are on the top and g_2, g_4 are on the bottom plate.

According to Lemma 4.2 there exist an oriented Hamiltonian path $\gamma_1 = (g_1, \xi x \eta)$ of $\mathcal{Q}_n^{top} - \{r_1\}$ connecting g_1 to g_3 and an oriented Hamiltonian path $\gamma_2 = (g_2, \mu x \theta)$ of $\mathcal{Q}_n^{bot} - \{r_2\}$ connecting g_2 to g_4 such that $g_1 \xi v = g_2 \mu$. The desired 2−path covering is $(g_1, \xi v \mu^R)$, $(g_3, \eta^R v \theta)$. □

In some proofs it is useful to be able to find Hamiltonian paths that pass through each element of a given set of vertices in such a way that the distance between two consecutive elements of that set along the path is at least 4. The following lemma gives a situation when that can be done. It will be used in the proofs of Lemma 4.5 and Lemma 5.12.

Lemma 4.4. *Let $n \geq 3$, $L = \{g_1, g_2, \ldots, g_{n-1}\}$ be a set of green vertices and r be a red vertex in $\mathcal{Q}_n$. Then there exists a Hamiltonian path in $\mathcal{Q}_n - \{r\}$ that connects g_1 to g_{n-1} in such a way that the distance along the path between any two vertices in L is at least 4.*

Proof. The proof is by induction. The statement is obvious for $n = 3$. Let $n \geq 3$ and $L = \{g_1, g_2, \ldots, g_{n-1}, g_n\}$ be a set of n green vertices and r be any red vertex in $\mathcal{Q}_{n+1}$. Produce plates in a way that $g_1 \in \mathcal{Q}_{n+1}^{top}$ and $g_n \in \mathcal{Q}_{n+1}^{bot}$. We can assume that $r \in \mathcal{Q}_{n+1}^{top}$ by renumbering g_1 and g_n, if necessary.

If g_1 is the only element of L in $\mathcal{Q}_{n+1}^{top}$ then use $[1, 1, 0, 1] = 2$ to produce a Hamiltonian path (g_1, ξ) of $\mathcal{Q}_{n+1}^{top} - \{r\}$ that connects g_1 to $g_2 x v$, where x is any letter different from v and such that $g_2 x v \neq g_1$. By the induction hypothesis there is a Hamiltonian path (g_2, η) of $\mathcal{Q}_{n+1}^{bot} - \{g_2 x\}$ that connects g_2 to g_n and such that the distance between any two different elements of L along this path is at least 4. The desired Hamiltonian path of $\mathcal{Q}_{n+1} - \{r\}$ for this case is $(g_1, \xi v x \eta)$.

If in addition to g_1 there is another element $g_i \in L \cap \mathcal{Q}_{n+1}^{top}$ (the total number of such elements cannot be more than $n-2$) then use the induction hypothesis to produce a Hamiltonian path (g_1, ξ) of $\mathcal{Q}_n^{top} - \{r\}$ that connects g_1 to g_i and such that the distance between any two elements of L along this path is at least 4. On the bottom plate there are at most $n-2$ elements of L. Therefore, there exists a letter x such that $g_i v x$ is not in L. By the induction hypothesis there is a Hamiltonian path $(g_i v x, \eta)$ of $\mathcal{Q}_{n+1}^{bot} - \{g_i v\}$ that connects $g_i v x$ to g_n and such that the distance between any two elements from L along the path is at least 4. The desired Hamiltonian path of $\mathcal{Q}_{n+1} - \{r\}$ for this case is $(g_1, \xi v x \eta)$. □

Lemma 4.5. ($[6] = 5$) *Let $n \geq 5$ and $\mathcal{F}$ be any neutral fault of mass 6 in $\mathcal{Q}_n$. Then $\mathcal{Q}_n - \mathcal{F}$ is Hamiltonian. The claim is not true if $n = 3$ or $n = 4$.*

Proof. Since $[2k] \geq k + 2$ for each integer $k \geq 0$, we have $[6] \geq 5$.

Let $n \geq 5$ and $\mathcal{F} = \{r_1, r_2, r_3, g_1, g_2, g_3\}$ be such that the first three vertices are red and the last three vertices are green. Produce two plates in such a way that r_1 and r_2 are on the top plate and r_3 is on the bottom plate. Then consider the four essentially different cases that depend on the distribution of the deleted green vertices on the plates.

Case 1. The three deleted green vertices are on the top plate.

Use $[4] = 4$ to find a Hamiltonian cycle (g_3, ξ) of $\mathcal{Q}_n^{top} - \{r_1, r_2, g_1, g_2\}$. Then use $[1, 1, 0, 1] = 2$ to find a Hamiltonian path $(g_3\varphi(\xi)v, \eta)$ of $\mathcal{Q}_n^{bot} - \{r_3\}$ that connects $g_3\varphi(\xi)v$ to $g_3\xi'v$. The desired Hamiltonian cycle of $\mathcal{Q}_n - \mathcal{F}$ for this case is $(g_3\varphi(\xi), v\eta v(\xi^R)'^*)$.

Case 2. g_1 and g_2 are on the top plate and g_3 is on the bottom plate.

Use $[4] = 4$ to produce a Hamiltonian cycle on $\mathcal{Q}_n^{top} - \{r_1, r_2, g_1, g_2\}$. Let a, b be two consecutive vertices along this cycle whose respective adjacent vertices on the bottom plate c, d are not deleted vertices. Use $[2, 0, 1, 0] = 4$ to connect c to d by a Hamiltonian path of $\mathcal{Q}_n^{bot} - \{r_3, g_3\}$. The desired Hamiltonian cycle of $\mathcal{Q}_n - \mathcal{F}$ for this case is obtained by removing the edge $\{a, b\}$ from the cycle constructed on the top plate and attaching to the resulting path by means of the bridges $\{a, c\}, \{b, d\}$ the path constructed on the bottom plate.

Case 3. g_1 is on the top plate and g_2 and g_3 are on the bottom plate.

Let g_4, g_5 be any two green non-deleted vertices on the top plate such that their respective adjacent vertices r_4, r_5 on the bottom plate are also non-deleted. Use $[3, 1, 0, 1] = 4$ to produce a Hamiltonian path of $\mathcal{Q}_n^{top} - \{r_1, r_2, g_1\}$ that connects g_4 to g_5. In the same way produce a Hamiltonian path of $\mathcal{Q}_n^{bot} - \{r_3, g_2, g_3\}$ that connects r_4 to r_5. The desired Hamiltonian cycle of $\mathcal{Q}_n - \mathcal{F}$ for this case is obtained by attaching the resulting paths to each other by means of the bridges $\{g_4, r_4\}, \{g_5, r_5\}$.

Case 4. The three green deleted vertices are on the bottom.

Use Lemma 4.4 to find a Hamiltonian path (g_1, ξ) of $\mathcal{Q}_n^{bot} - \{r_3\}$ that connects g_1 to g_3 and such that $\xi = \eta\theta$, with $g_1\eta = g_2$, and both η and θ have length at least four. Then use $[2, 2, 0, 2] = 4$ to produce a 2−path covering of $\mathcal{Q}_n^{top} - \{r_1, r_2\}$ with paths $(g_1\varphi(\eta)v, \mu), (g_1\eta'v, \nu)$ connecting $g_1\varphi(\eta)v$ to $g_2\varphi(\theta)v$ and $g_1\eta'v$ to $g_2\theta'v$, respectively. The desired Hamiltonian cycle of $\mathcal{Q}_n - \mathcal{F}$ for this case is $(g_1\varphi(\eta), v\mu v\theta'^* v\nu^R v(\eta^R)'^*)$. □

Lemma 4.6. ($[4, 0, 1, 0] = 5$) *Let $n \geq 5$, r, r_1, r_2 be three distinct red vertices and g, g_1, g_2 be three distinct green vertices in $\mathcal{Q}_n$. Then there exists a Hamiltonian path of $\mathcal{Q}_n - \{r_1, r_2, g_1, g_2\}$ that connects r to g. The claim is not true if $n = 3$ or $n = 4$.*

Proof. Let $r=(0,1,0,0)$, $r_1=(1,0,0,0)$, $r_2=(1,1,1,0)$ and $g=(1,0,0,1)$, $g_1=(1,1,1,1)$, $g_2=(0,0,1,1)$ be vertices in $\mathcal{Q}_4$. Then one can verify directly that a Hamiltonian path of $\mathcal{Q}_4-\{r_1,r_2,g_1,g_2\}$ connecting r to g does not exist.

Let $n\geq 5$. Choose two plates that separate the deleted red vertices and consider the six essentially different cases depending on the distribution of the green deleted vertices and the terminals on the plates. We can assume that r_1 is the deleted red vertex on the top plate and r_2 is the deleted red vertex on the bottom plate.

Case A. The two deleted green vertices are on the top plate.

Subcase A1. The two terminals are on the top plate.

Use $[2,0,1,0]=4$ to produce a Hamiltonian path (r,ξ) of $\mathcal{Q}_n^{top}-\{r_1,g_1\}$ that connects r to g and let $\xi=\mu\eta$, with $r\mu=g_2$. Use $[1,1,0,1]=2$ to produce a Hamiltonian path $(r\mu'v,\theta)$ of $\mathcal{Q}_n^{bot}-\{r_2\}$ that connects $r\mu'v$ to $r\mu\varphi(\eta)v$. The desired Hamiltonian path of $\mathcal{Q}_n-\{r_1,r_2,g_1,g_2\}$ for this case is $(r,\mu'v\theta v\eta^*)$.

Subcase A2. g is on the top plate and r is on the bottom plate.

Let r_3 be a red vertex on the top plate at a distance at least three away from g_2. Use $[2,0,1,0]=4$ to produce a Hamiltonian path (g,ξ) of $\mathcal{Q}_n^{top}-\{r_1,g_1\}$ that connects g to r_3. Let $\xi=\mu\eta$, with $g\mu=g_2$ and η of length at least three. Use $[1,1,1,1]=4$ to produce a $2-$path covering of $\mathcal{Q}_n^{bot}-\{r_2\}$ with paths $(g\mu'v,\theta),(r,\nu)$ connecting $g\mu'v$ to $g\mu\varphi(\eta)v$ and r to r_3v, respectively. The desired Hamiltonian path of $\mathcal{Q}_n-\{r_1,r_2,g_1,g_2\}$ for this case is $(g,\mu'v\theta v\eta^*v\nu^R)$.

Subcase A3. r is on the top plate and g is on the bottom plate.

Let r_3 be a red vertex on the top plate which is not adjacent to g. Use $[3,1,0,1]=4$ to produce a Hamiltonian path (r,ξ) of $\mathcal{Q}_n^{top}-\{r_1,g_1,g_2\}$ that connects r to r_3. Use $[1,1,0,1]=2$ to produce a Hamiltonian path (r_3v,μ) of $\mathcal{Q}_n^{bot}-\{r_2\}$ connecting r_3v to g. Then the desired Hamiltonian path of $\mathcal{Q}_n-\{r_1,r_2,g_1,g_2\}$ for this case is $(r,\xi v\mu)$.

Subcase A4. r and g are both on the bottom plate.

Let r_3 and r_4 be two red vertices on the top plate such that r_3v and r_4v are different from g. Use $[3,1,0,1]=4$ to produce a Hamiltonian path (r_3,ξ) of $\mathcal{Q}_n^{top}-\{r_1,g_1,g_2\}$ that connects r_3 to r_4. Use $[1,1,1,1]=4$ to produce a $2-$path covering of $\mathcal{Q}_n^{bot}-\{r_2\}$ with paths (r,η) and (r_4v,μ) connecting r to r_3v and r_4v to g, respectively. The desired Hamiltonian path of $\mathcal{Q}_n-\{r_1,r_2,g_1,g_2\}$ for this case is $(r,\eta v\xi v\mu)$.

Case B. Each plate contains one deleted green vertex. We can assume that g_1 is on the top plate and g_2 is on the bottom plate.

Subcase B1. The two terminals are on the top plate.

Use $[2,0,1,0]=4$ to produce a Hamiltonian path (r,ξ) of $\mathcal{Q}_n^{top}-\{r_1,g_1\}$ that connects r to g. Since $n-1\geq 4$ there exist words μ and η and a letter x such that $\xi=\mu x\eta$ with neither $r\mu v$ nor $r\mu xv$ being a deleted vertex. Use again $[2,0,1,0]=4$ to produce a Hamiltonian path $(r\mu v,\zeta)$ of $\mathcal{Q}_n^{bot}-\{r_2,g_2\}$ that connects $r\mu v$ to $r\mu xv$. The desired Hamiltonian path of $\mathcal{Q}_n-\{r_1,r_2,g_1,g_2\}$ for this case is $(r,\mu v\zeta v\eta)$.

Subcase B2. g is on the top plate and r is on the bottom plate.

Let r_3 be any red vertex on the top plate such that $r_3v \neq g_2$. Use $[2,0,1,0] = 4$ to produce a Hamiltonian path (g,ξ) of $\mathcal{Q}_n^{top} - \{r_1, g_1\}$ connecting g to r_3. Use again $[2,0,1,0] = 4$ to produce a Hamiltonian path (r_3v, η) of $\mathcal{Q}_n^{bot} - \{r_2, g_2\}$ connecting r_3v to r. The desired Hamiltonian path in $\mathcal{Q}_n - \{r_1, r_2, g_1, g_2\}$ for this case is $(g, \xi v\eta)$. □

Lemma 4.7. ($[0,0,3,0] = 5$) *Let $n \geq 5$, r_1, r_2, r_3 be three distinct red vertices and g_1, g_2, g_3 be three distinct green vertices in $\mathcal{Q}_n$. Then there exists a $3-$path covering of $\mathcal{Q}_n$ with paths γ_i connecting r_i to g_i for $i = 1, 2, 3$. The claim is not true if $n = 3$ or $n = 4$.*

Proof. Let $r_1 = (0,0,0,0)$, $r_2 = (0,1,0,1)$, $r_3 = (0,1,1,0)$, $g_1 = (0,1,1,1)$, $g_2 = (0,0,1,0)$, and $g_3 = (0,0,0,1)$ be vertices in $\mathcal{Q}_4$. Then it is not difficult to verify that a $3-$path covering of $\mathcal{Q}_4$ with paths γ_i connecting r_i to g_i for $i = 1, 2, 3$ does not exist (see also [11, Fig.1]).

Let $n \geq 5$. Choose two plates to split the deleted red vertices such that r_1 and r_2 are on $\mathcal{Q}_n^{top}$ and r_3 is on $\mathcal{Q}_n^{bot}$. There are five substantially different cases depending on the distribution of the green terminals on the plates.

Case 1. The three green terminals are on the top plate.

Use $[0,0,2,0] = 2$ to produce a path covering (r_1,ξ), (r_2,η) of $\mathcal{Q}_n^{top}$ that connects r_1 to g_1 and r_2 to g_2. Without loss of generality we may assume that g_3 lies on the path between r_2 and g_2. Let $\eta = \mu\theta$, where $r_2\mu = g_3$.

If $g_3v \neq r_3$ then use $[0,0,0,2] = 4$ to produce a path covering $(r_2\mu'v, \nu)$, (g_3v, ζ) of $\mathcal{Q}_n^{bot}$ that connects $r_2\mu'v$ to $g_2(\theta^R)'v$ and g_3v to r_3. The desired $3-$path covering for this case is (r_1,ξ), $(r_2, \mu'v\nu v\theta^*)$, $(r_3, \zeta^R v)$.

If $g_3v = r_3$ then use $[1,1,0,1] = 2$ to produce a Hamiltonian path $(r_2\mu'v, \nu)$ of $\mathcal{Q}_n^{bot} - \{r_3\}$ that connects $r_2\mu'v$ to $g_2(\theta^R)'v$. The desired $3-$path covering for this case is (r_1,ξ), $(r_2, \mu'v\nu v\theta^*)$, (r_3, r_3v).

Case 2. Two green terminals are on the top plate and one is on the bottom plate.

If the green terminal on the bottom plate is g_3 then use $[0,0,2,0] = 2$ to produce a $2-$path covering of $\mathcal{Q}_n^{top}$ connecting r_1 to g_1 and r_2 to g_2 and use $[0,0,1,0] = 1$ to produce a Hamiltonian path of $\mathcal{Q}_n^{bot}$ connecting r_3 to g_3.

Now, assume that g_1 and g_3 are on the top plate and g_2 is on the bottom plate.

If $r_2v \neq g_2$ and $g_3v \neq r_3$ then use $[2,0,1,0] = 4$ to find a Hamiltonian path (r_1,ξ) of $\mathcal{Q}_n^{top} - \{r_2, g_3\}$ connecting r_1 to g_1 and use $[0,0,0,2] = 4$ to produce a $2-$path covering (r_2v, η), (r_3, ζ) of $\mathcal{Q}_n^{bot}$ that connects r_2v to g_2 and r_3 to g_3v. The desired $3-$path covering for this case is (r_1,ξ), $(r_2, v\eta)$, $(r_3, \zeta v)$.

Let $r_2v \neq g_2$ and $g_3v = r_3$ (the case $r_2v = g_2$ and $g_3v \neq r_3$ is symmetrical). Use $[2,0,1,0] = 4$ to find a Hamiltonian path (r_1,ξ) of $\mathcal{Q}_n^{top} - \{r_2, g_3\}$ connecting r_1 to g_1 and use $[1,1,0,1] = 2$ to produce a Hamiltonian path (r_2v, η) of $\mathcal{Q}_n^{bot} - \{r_3\}$ that connects r_2v to g_2. The desired $3-$path covering for this case is (r_1,ξ), $(r_2, v\eta)$, (r_3, v).

Finally, let $r_2v = g_2$ and $g_3v = r_3$. Use $[2,0,1,0] = 4$ to find a Hamiltonian path (r_1, ξ) of $\mathcal{Q}_n^{top} - \{r_2, g_3\}$ connecting r_1 to g_1. Clearly, the length of the path (r_1, ξ) is more than 1. Use $[2,0,1,0] = 4$ to find a Hamiltonian path (r_1v, η) of $\mathcal{Q}_n^{bot} - \{r_3, g_2\}$ connecting r_1v to $r_1\varphi(\xi)v$. The desired 3−path covering for this case is $(r_1, v\eta v\xi^*)$, (r_2, v), (r_3, v).

Case 3. g_3 is on the top plate and the other two green terminals are on the bottom plate.

If $r_3v = g_3$ then use $[1,1,0,1] = 2$ to find a Hamiltonian path (r_1, ξ) of $\mathcal{Q}_n^{top} - \{g_3\}$ that connects r_1 to r_2. Let $\xi = \mu x\eta$, with neither $r_1\mu v$ nor $r_1\mu xv$ being a prescribed end. On the bottom plate use $[1,1,1,1] = 4$ to produce a 2−path covering $(r_1\mu v, \theta)$, $(r_1\mu xv, \zeta)$ of $\mathcal{Q}_n^{bot} - \{r_3\}$ connecting $r_1\mu v$ to g_1 and $r_1\mu xv$ to g_2, respectively. The desired 3−path covering for this case is $(r_1, \mu v\theta)$, $(r_2, \eta^R v\zeta)$, (r_3, v).

If $r_3v \neq g_3$ use Corollary 3.10 to produce a 2−path covering (g_3, ξ) and (r_1, η) of the top plate with the first path connecting g_3 to r_3v and the second path of length at least 8 connecting r_1 to r_2. Let $\eta = \mu x\theta$, with neither $r_1\mu v$ nor $r_1\mu xv$ being a prescribed end. Use $[1,1,1,1] = 4$ to produce a 2−path covering $(r_1\mu v, \nu)$, $(r_1\mu xv, \zeta)$ of $\mathcal{Q}_n^{bot} - \{r_3\}$ connecting $r_1\mu v$ to g_1 and $r_1\mu xv$ to g_2, respectively. The desired 3−path covering of $\mathcal{Q}_n$ for this case is $(r_1, \mu v\nu)$, $(r_2, \theta^R v\zeta)$, $(r_3, v\xi^R)$.

Case 4. Either g_1 or g_2 is on the top plate and the other two green terminals are on the bottom plate.

Without loss of generality we can assume that g_1 is the green terminal on the top plate. Let g_4 be any green vertex on the top plate such that g_4v is not a terminal vertex. Use $[0,0,2,0] = 2$ to find a 2−path covering (r_1, ξ), (r_2, η) of $\mathcal{Q}_n^{top}$ that connects r_1 to g_1 and r_2 to g_4, and a 2−path covering (r_3, μ), $(r_2\eta v, \nu)$ of $\mathcal{Q}_n^{bot}$ that connects r_3 to g_3 and $r_2\eta v$ to g_2. The desired 3−path covering for this case is (r_1, ξ), $(r_2, \eta v\nu)$, (r_3, ν).

Case 5. All the green terminals are on the bottom plate.

Let $r_4 = g_1x$ be any vertex on the bottom plate adjacent to g_1 and different from r_3. Use $[2,0,1,0] = 4$ to produce a Hamiltonian path (g_2, ξ) of $\mathcal{Q}_n^{bot} - \{g_1, r_4\}$ that connects g_2 to r_3. Let $\xi = \mu\eta$, with $g_2\mu = g_3$. Use $[0,0,2,0] = 2$ to produce a 2−path covering (r_4v, θ), $(g_2\mu' v, \zeta)$ of $\mathcal{Q}_n^{top}$ connecting r_4v to r_1 and $g_2\mu' v$ to r_2, respectively. The desired 3−path covering of $\mathcal{Q}_n$ for this case is $(g_1, xv\theta)$, $(g_2, \mu' v\zeta)$, (g_3, η). □

5. Some general results

Let $\mathcal{G}$ be a graph and v be a vertex in $\mathcal{G}$. We denote by $\mathcal{N}(v)$ the set of vertices adjacent to v in $\mathcal{G}$. If A is a subset of the set of vertices of $\mathcal{G}$ then the set $\mathcal{N}(A) = \bigcup_{v\in A} \mathcal{N}(v)$ is called the set of neighbors of A.

As usual, if X is a set, $|X|$ denotes the cardinality of X.

Proposition 5.1. *Let $A \subset \mathcal{N}(r)$ for some vertex r in $\mathcal{Q}_n$. Then* $|\mathcal{N}(A)| = 1 + n|A| - \frac{|A|(|A|+1)}{2}$.

Proof. Obviously $r \in \mathcal{N}(A)$. Any pair of elements $g_1, g_2 \in A$ has exactly two neighbors in common one of which is the root r, and the other is different for different pairs. It follows that

$$|\mathcal{N}(A)| = 1 + (n-1) + (n-2) + \cdots + (n - |A|). \qquad \square$$

The following lemma is a particular case of an isoperimetric inequality for the hypercube. See [1, Theorem 7.3] for a more general statement and a discussion of several proofs available in the literature. Here we just state and prove what we need in the sequel.

Lemma 5.2. *Let k and n be positive integers such that $1 \leq k \leq n$ and let A be a set of green vertices in $\mathcal{Q}_n$ of cardinality k. Then*

$$|\mathcal{N}(A)| \geq 1 + (n-1) + \cdots + (n-k) = 1 + kn - \frac{k(k+1)}{2},$$

with equality if and only if $A \subset \mathcal{N}(r)$ for some red vertex r.

Proof. The statement is obvious for all pairs k, n with $1 \leq k \leq 2$ and $k \leq n$. Let N be a positive integer greater than 2 such that the statement is true for all pairs k, n with $1 \leq k \leq n$ and $n < N$. We shall prove that the statement is also true for all pairs k, N with $1 \leq k \leq N$.

We split $\mathcal{Q}_N$ into two plates such that $1 \leq l = |A \cap Q_N^{bot}| \leq m = |A \cap \mathcal{Q}_N^{top}| \leq N - 1$. Let $A^{top} = A \cap \mathcal{Q}_N^{top}$ and $A^{bot} = A \cap \mathcal{Q}_N^{bot}$. Each element of A^{top} has exactly one neighbor in $\mathcal{Q}_N^{bot}$. Therefore, by Proposition 5.1 and the induction hypothesis,

$$|\mathcal{N}(A^{top})| \geq 1 + [(N-1) - 1] + \cdots + [(N-1) - m] + m$$

$$= 1 + (N-1) + \cdots + (N-m),$$

with equality throughout if and only if there exists $r \in \mathcal{Q}_N^{top}$ such that $A^{top} \subset \mathcal{N}(r)$.

Similarly, let s be the number of elements of $\mathcal{N}(A^{bot})$ that are in the top plate but not in $\mathcal{N}(A^{top})$. Then

$$|\mathcal{N}(A^{bot}) \setminus \mathcal{N}(A^{top})| \geq -m + 1 + [(N-1) - 1] + \cdots + [(N-1) - l] + s$$

$$\geq [(N-m) - 1] + \cdots + [(N-m) - l],$$

with equality throughout if and only if $l = 1$ and $s = 0$. It follows that $|\mathcal{N}(A)| \geq 1 + N - 1 + \cdots + N - k$ with equality if and only if there exists a vertex $r \in \mathcal{Q}_N$ such that $A \subset \mathcal{N}(r)$. $\square$

Lemma 5.3. *Let M, C, N, O be nonnegative integers with C, O, and M of the same parity, $C \leq M$, $O \geq C$, and $N \geq 1$. Let also k be a positive integer such that*

$$kN + 1 - \binom{N+1}{2} > \frac{M+C}{2} + N + O. \tag{1}$$

Then, $k \in \mathcal{A}_{M+1,C+1,N-1,O+1}$ implies $k \in \mathcal{A}_{M,C,N,O}$.

Proof. Let $k \in \mathcal{A}_{M+1,C+1,N-1,O+1}$. This means that if $n \geq k$ then for every fault $\mathcal{F}$ of mass $M+1$ and charge $C+1$ in $\mathcal{Q}_n$ one can freely prescribe ends for a path covering of $\mathcal{Q}_n - \mathcal{F}$ with $N-1$ neutral paths and $O+1$ charged paths. Consider an arbitrary fault $\mathcal{F}$ of mass M and charge C in $\mathcal{Q}_k$, and a set $\mathcal{E}$ of pairs of vertices that contains N neutral pairs and O charged pairs, and is in balance with $\mathcal{F}$.

Without loss of generality we may assume that in $\mathcal{F}$ there are at least as many red vertices as there are green vertices. It is easy to see that the number of the deleted green vertices is $\frac{M-C}{2}$, and that the number of paths with green terminals at both ends is $\frac{O+C}{2}$. Thus, the quantity $\frac{M+C}{2} + N + O$ is the total number of green vertices that are either deleted vertices or terminal vertices.

The number of red terminals in neutral pairs is obviously N. By Lemma 5.2 the number of green vertices that are adjacent to at least one red terminal in a neutral pair is at least $kN + 1 - \binom{N+1}{2}$. Therefore, inequality (1) guarantees the existence of a neutral pair $(r, g) \in \mathcal{E}$ and a green vertex $g' = rx$ that is neither a deleted vertex nor a terminal vertex. The fault $\mathcal{F}' = \mathcal{F} \cup \{r\}$ has mass $M+1$ and charge $C+1$. The set of pairs of vertices $\mathcal{E}'$ obtained from $\mathcal{E}$ by replacing the pair (r, g) with the pair (g', g) is in balance with $\mathcal{F}'$ and contains $N-1$ neutral pairs and $O+1$ charged pairs. Therefore, there exists an $N+O-$path covering of $\mathcal{Q}_k - \mathcal{F}$ whose set of pairs of end vertices coincide with $\mathcal{E}'$. One of the paths in this covering is of the form (g, ξ) with $g\xi = g'$. If we replace this path with the path $(g, \xi x)$ that connects g to r we obtain an $N+O-$path covering of $\mathcal{Q}_k - \mathcal{F}$ whose set of pairs of end vertices coincides with $\mathcal{E}$. So, we proved that for every fault $\mathcal{F}$ of mass M and charge C in $\mathcal{Q}_k$ one can freely prescribe ends for a path covering of $\mathcal{Q}_k - \mathcal{F}$ with N neutral paths and O charged paths. Finally, if $n \geq k$ then 1) $nN + 1 - \binom{N+1}{2} > \frac{M+C}{2} + N + O$, and 2) $n \in \mathcal{A}_{M+1,C+1,N-1,O+1}$. Therefore, the argument that we applied to k can be applied to n as well. This shows that if $n \geq k$ then for every fault $\mathcal{F}$ of mass M and charge C in $\mathcal{Q}_n$ one can freely prescribe ends for a path covering of $\mathcal{Q}_n - \mathcal{F}$ with N neutral paths and O charged paths. Consequently $k \in \mathcal{A}_{M,C,N,O}$. □

Lemma 5.4. *Let M, C, N, O be nonnegative integers with C, O, and M of the same parity, $C \leq M$, and $O > C$. Let also k be a positive integer such that*

$$k(O-C) + 1 - \binom{O-C+1}{2} > \frac{M+C}{2} + N + O. \tag{2}$$

Then, $k \in \mathcal{A}_{M+1,C+1,N+1,O-1}$ *implies* $k \in \mathcal{A}_{M,C,N,O}$.

Proof. The proof is similar to the proof of Lemma 5.3. The only difference is that in (2), instead of N, we use the number $O-C$ that represents the number of red terminals in the charged paths. □

Lemma 5.5. *Let M, C, N, O be nonnegative integers with C, O, and M of the same parity, $C \le M$, $O \ge C$, and $C \ge 1$. Let also k be a positive integer such that*

$$k(O+C) + 1 - \binom{O+C+1}{2} > \frac{M-C}{2} + N + O. \tag{3}$$

Then, $k \in \mathcal{A}_{M+1,C-1,N+1,O-1}$ *implies* $k \in \mathcal{A}_{M,C,N,O}$.

Proof. The proof is similar to the proof of Lemma 5.3. The difference is that in the left-hand side of (3), instead of N, we use the number $O+C$ that represents the number of green terminals in the charged paths and the right-hand side part $\frac{M-C}{2} + N + O$ represents the number of red vertices that are either in $\mathcal{F}$ or are terminals. □

Lemma 5.6. $[4,2,0,2] = [3,1,1,1] = 5$ *and* $[2,0,2,0] = 4$.

Proof. It follows from Lemma 5.3 that if $5 \in \mathcal{A}_{4,2,0,2}$ then 5 is in $\mathcal{A}_{3,1,1,1}$ and in $\mathcal{A}_{2,0,2,0}$. Lemma A.1, proved in Appendix A, states that we can freely prescribe two neutral pairs of terminals for a 2−path covering of $\mathcal{Q}_4 - \mathcal{F}$ for any neutral fault of mass 2. Therefore, to prove the current lemma, it is sufficient to show that $5 \in \mathcal{A}_{4,2,0,2}$, $4 \notin \mathcal{A}_{3,1,1,1}$ (and therefore, according to Lemma 5.3, $4 \notin \mathcal{A}_{4,2,0,2}$), and that $3 \notin \mathcal{A}_{2,0,2,0}$.

Here is a counterexample showing that $3 \notin \mathcal{A}_{2,0,2,0}$. Let $n = 3$, $r_1 = (1,0,0)$, $g_1 = (0,1,1)$, $r_2 = (0,1,0)$, $g_2 = (1,0,1)$, and $\mathcal{F} = \{(0,0,0),(1,1,1)\}$. Then, a 2−path covering of $\mathcal{Q}_3 - \mathcal{F}$ that connects r_1 to g_1 and r_2 to g_2 does not exist.

The following counterexample shows that $4 \notin \mathcal{A}_{3,1,1,1}$ (see also the discussion after Conjecture 6.4).

Let $n = 4$, $\mathcal{F} = \{(0,0,0,0),(0,1,0,1),(0,1,1,1)\}$, $r_1 = (1,1,0,0)$, $g_1 = (1,0,0,0), g_2 = (0,0,1,0)$, and $g_3 = (1,1,1,0)$. Then, a 2−path covering of $\mathcal{Q}_4 - \mathcal{F}$ that connects r_1 to g_1 and g_2 to g_3 does not exist.

We now prove that $5 \in \mathcal{A}_{4,2,0,2}$. Let $n \ge 5$. We can assume that $\mathcal{F} = \{r_1, r_2, r_3, g\}$ with r_1, r_2, r_3 being red and g being a green vertex. Let $\mathcal{E} = \{(g_1,g_2),(g_3,g_4)\}$ be the set of pairs of green end vertices. We are looking for 2−path coverings of $\mathcal{Q}_n - \mathcal{F}$ with paths that connect g_1 to g_2 and g_3 to g_4. We split $\mathcal{Q}_n$ into two plates with two red vertices in the top plate, say r_1 and r_2, and r_3 in the bottom plate. Then we consider a group of cases when the green deleted vertex g is on the top plate and another group of cases when the green deleted vertex is on the bottom plate. The cases within each group depend on the distribution of the green terminals on the plates.

Case A. The green deleted vertex is on the top plate.

Subcase A1. All the green terminals are on the top plate.

Let (g_1, ξ) be a Hamiltonian path on $\mathcal{Q}_n^{top} - \{r_1, r_2, g\}$ that connects g_1 to g_2. Such path exists since $[3, 1, 0, 1] = 4$. Let $\xi = \eta\theta\mu$ with $g_1\eta = g_3$ and $g_1\eta\theta = g_4$, where g_3, g_4 are renumbered, if necessary. Let $(g_1\xi' v, \zeta)$ be a Hamiltonian path on $\mathcal{Q}_n^{bot} - \{r_3\}$ that connects $g_1\xi' v$ to $g_2(\mu^R)'v$. Such path exists since $[1, 1, 0, 1] = 2$. The desired $2-$path covering of $\mathcal{Q}_n - \mathcal{F}$ for this case is $(g_1, \xi' v\zeta v\mu^*), (g_3, \theta)$.

Subcase A2. g_1, g_2, g_3 are on the top plate and g_4 is on the bottom plate.

Let (g_1, ξ) be a Hamiltonian path on $\mathcal{Q}_n^{top} - \{r_1, r_2, g\}$ that connects g_1 to g_3. Such path exists since $[3, 1, 0, 1] = 4$. Let $\xi = \eta\theta$ with $g_1\eta = g_2$. Let $(g_3(\theta^R)'v, \zeta)$ be a Hamiltonian path on $\mathcal{Q}_n^{bot} - \{r_3\}$ that connects $g_3(\theta^R)'v$ to g_4. Such path exists since $[1, 1, 0, 1] = 2$. The desired $2-$path covering of $\mathcal{Q}_n - \mathcal{F}$ for this case is $(g_1, \eta), (g_3, (\theta^R)'v\zeta)$.

Subcase A3. g_1, g_2 are on the top plate and g_3, g_4 are on the bottom plate.

We simply connect g_1 to g_2 by a Hamiltonian path of $\mathcal{Q}_n^{top} - \{r_1, r_2, g\}$ and g_3 to g_4 by a Hamiltonian path of $\mathcal{Q}_n^{bot} - \{r_3\}$. That produces the desired $2-$path covering of $\mathcal{Q}_n - \mathcal{F}$ for this case.

Subcase A4. g_1, g_3 are on the top plate and g_2, g_4 are on the bottom plate.

Let (g_1, ξ) be a Hamiltonian path on $\mathcal{Q}_n^{top} - \{r_1, r_2, g\}$ that connects g_1 to g_3. Such path exists since $[3, 1, 0, 1] = 4$. We can find words η, θ, and a letter x such that $\xi = \eta x\theta$, and neither $g_1\eta v$ nor $g_1\eta x v$ is a deleted vertex or a terminal. Let $(g_1\eta v, \mu), (g_1\eta x v, \nu)$ be a $2-$path covering of $\mathcal{Q}_n^{bot} - \{r_3\}$ that connects $g_1\eta v$ to g_2 and $g_1\eta x v$ to g_4. Such path covering exists since $[1, 1, 1, 1] = 4$. The desired $2-$path covering of $\mathcal{Q}_n - \mathcal{F}$ for this case is $(g_1, \eta v\mu), (g_3, \theta^R v\nu)$.

Subcase A5. g_1 is on the top plate and g_2, g_3, g_4 are on the bottom plate.

Let $r \neq r_3$ be a red vertex on the bottom plate such that $rv \neq g_1, g$. Let $(g_2, \eta), (g_3, \theta)$ be a $2-$path covering of $\mathcal{Q}_n^{bot} - \{r_3\}$ that connects g_2 to r and g_3 to g_4. Such path covering exists since $[1, 1, 1, 1] = 4$. Let (g_1, μ) be a Hamiltonian path of $\mathcal{Q}_n^{top} - \{r_1, r_2, g\}$ that connects g_1 to rv. The desired $2-$path covering of $\mathcal{Q}_n - \mathcal{F}$ for this case is $(g_1, \mu v\eta^R), (g_3, \theta)$.

Subcase A6. All the green terminals are on the bottom plate.

First we assume that either g_3 or g_4 (or, equivalently, g_1 or g_2) is not adjacent to gv. Without loss of generality we can assume that g_3 is at distance at least three from gv and let x be a letter such that $g_2xv \neq g$. Let (g_1, ξ) be a Hamiltonian path of $\mathcal{Q}_n^{bot} - \{r_3, g_2, g_2x\}$ that connects g_1 to g_4. Such path exists since $[3, 1, 0, 1] = 4$. Then $\xi = \eta\theta$ with $g_1\eta = g_3$. Observe that our assumption on g_3 guarantees that $g_1\eta' v \neq g$. Let $(g_1\eta' v, \zeta)$ be a Hamiltonian path on $\mathcal{Q}_n^{top} - \{r_1, r_2, g\}$ that connects $g_1\eta' v$ to g_2xv. Such path exists since $[3, 1, 0, 1] = 4$. The desired $2-$path covering of $\mathcal{Q}_n - \mathcal{F}$ for this case is $(g_1, \eta' v\zeta v x), (g_3, \theta)$.

Now let us assume that $gv = r_3$ and all the vertices g_1, g_2, g_3 and g_4 are adjacent to gv. Then we can use the same construction as in the previous case to find the desired 2−path covering. In this case the requirement one of the green terminals to be at distance three from gv is not necessary since $gv = r_3$.

Finally, let us assume that $gv \neq r_3$ and g_3 and g_4 are adjacent to gv. This means that there exist letters x, y such that $g_3x = g_4y = gv$. Let (g_1, ξ) be a Hamiltonian cycle in $\mathcal{Q}_n^{bot} - \{r_3, g_3, g_4, gv\}$. Such cycle exists since $[4] = 4$. Then $\xi = \eta\theta$ with $g_1\eta = g_2$. Let $(g_1\eta' v, \zeta)$ be a Hamiltonian path of $\mathcal{Q}_n^{top} - \{r_1, r_2, g\}$ that connects $g_1\eta' v$ to $g_1\xi' v$. Such path exists since $[3, 1, 0, 1] = 4$. The desired 2−path covering of $\mathcal{Q}_n - \mathcal{F}$ for this case is $(g_1, \eta' v\zeta v(\theta')^R), (g_3, xy)$.

Case B. The green deleted vertex is on the bottom plate.

Subcase B1. All the green terminals are on the top plate.

Let $(g_1, \xi), (g_3, \eta)$ be a 2−path covering of $\mathcal{Q}_n^{top} - \{r_1, r_2\}$ that connects g_1 to g_2 and g_3 to g_4. Such path covering exists since $[2, 2, 0, 2] = 4$. Without loss of generality we can assume that the word ξ is not shorter than the word η. Therefore, there exist words μ, ν and a letter x such that $\xi = \mu x \nu$ with neither $g_1\mu v$ nor $g_1\mu x v$ being a deleted vertex. Let $(g_1\mu v, \zeta)$ be a Hamiltonian path of $\mathcal{Q}_n^{bot} - \{r_3, g\}$ that connects $g_1\mu v$ to $g_1\mu x v$. Such path exists since $[2, 0, 1, 0] = 4$. The desired 2−path covering of $\mathcal{Q}_n - \mathcal{F}$ for this case is $(g_1, \mu v\zeta v\nu), (g_3, \eta)$.

Subcase B2. g_1, g_2, g_3 are on the top plate and g_4 is on the bottom plate.

Let g_5 be a green vertex on the top plate such that g_5v is not a deleted vertex. Let $(g_1, \xi), (g_3, \eta)$ be a 2−path covering of $\mathcal{Q}_n^{top} - \{r_1, r_2\}$ that connects g_1 to g_2 and g_3 to g_5. Such path covering exists since $[2, 2, 0, 2] = 4$. Let (g_5v, ζ) be a Hamiltonian path of $\mathcal{Q}_n^{bot} - \{r_3, g\}$ that connects g_5v to g_4. Such path exists since $[2, 0, 1, 0] = 4$. The desired 2−path covering of $\mathcal{Q}_n - \mathcal{F}$ for this case is $(g_1, \xi), (g_3, \eta v\zeta)$.

Subcase B3. g_1, g_2 are on the top plate and g_3, g_4 are on the bottom plate.

Since $n \geq 4$ we can find words η, θ of length greater than three such that $(g_3, \eta\theta)$ is a Hamiltonian cycle of $\mathcal{Q}_n^{bot} - \{r_3, g\}$ with $g_3\eta = g_4$ (Lemma 3.14). For at least one of the four pairs of green vertices $(g_3\varphi(\eta)v, g_3\eta' v)$, $(g_3\varphi(\eta)v, g_4\varphi(\theta)v)$, $(g_3\eta' v, g_4\theta' v)$, $(g_4\varphi(\theta)v, g_4\theta' v)$ the two elements in the pair are not terminals on the top plate.

Assume that neither $g_3\varphi(\eta)v$ nor $g_3\eta' v$ is a terminal vertex. Let (g_1, μ), (g_2, ν) be a 2−path covering of $\mathcal{Q}_n^{top} - \{r_1, r_2\}$ that connects g_1 to $g_3\varphi(\eta)v$ and g_2 to $g_3\eta' v$. Such path covering exists since $[2, 2, 0, 2] = 4$. The desired 2−path covering of $\mathcal{Q}_n - \mathcal{F}$ for this case is $(g_1, \mu v\eta'^{*} v\nu^R)$, (g_3, θ^R).

The case when neither $g_4\varphi(\theta)v$ nor $g_4\theta' v$ is a terminal vertex is equivalent to the previous case.

Assume now that neither $g_3\varphi(\eta)v$ nor $g_4\varphi(\theta)v$ is a terminal vertex. Let $(g_1, \mu), (g_4\varphi(\theta)v, \nu)$ be a 2−path covering of $\mathcal{Q}_n^{top} - \{r_1, r_2\}$ that connects g_1 to g_2 and $g_4\varphi(\theta)v$ to $g_3\varphi(\eta)v$. Such path covering exists since $[2, 2, 0, 2] = 4$. The desired 2−path covering of $\mathcal{Q}_n - \mathcal{F}$ for this case is $(g_1, \mu), (g_3, (\theta^R)' v\nu v\eta^{*})$.

The case when neither $g_3\eta' v$ nor $g_4\theta' v$ is a terminal vertex is equivalent to the previous case.

Subcase B4. g_1, g_3 are on the top plate and g_2, g_4 are on the bottom plate.

Let x be a letter such that $g_2x \neq r_3$ and $g_2xv \neq g_1, g_3$. Such letter exists since the dimension of the plates is greater than or equal to 4. Let (g_4, ξ) be a Hamiltonian cycle of $\mathcal{Q}_n^{bot} - \{r_3, g, g_2, g_2x\}$. Such cycle exists since $[4] = 4$. We can also assume that $g_4\xi' v \neq g_1$ by replacing ξ with ξ^R, if necessary.

Assume that $g_4\xi' v = g_3$. Let (g_1, μ) be a Hamiltonian path of $\mathcal{Q}_n^{top} - \{r_1, r_2, g_3\}$ that connects g_1 to g_2xv. Such path exists since $[3, 1, 0, 1] = 4$. The desired $2-$path covering of $\mathcal{Q}_n - \mathcal{F}$ for this case is $(g_1, \mu vx)$, $(g_4, \xi' v)$.

Finally, if $g_4\xi' v \neq g_3$ we proceed as follows. Let $(g_1, \mu), (g_3, \nu)$ be a $2-$path covering of $\mathcal{Q}_n^{top} - \{r_1, r_2\}$ that connects g_1 to g_2xv and g_3 to $g_4\xi' v$. Such path covering exists since $[2, 2, 0, 2] = 4$. The desired $2-$path covering of $\mathcal{Q}_n - \mathcal{F}$ for this case is $(g_1, \mu vx), (g_3, \nu v(\xi')^R)$.

Subcase B5. g_1 is on the top plate and g_2, g_3, g_4 are on the bottom plate.

Let x be a letter different from v such that $g_2x \neq r_3$ and $g_2xv \neq g_1$. Let (g_3, ξ) be a Hamiltonian cycle of $\mathcal{Q}_n^{bot} - \{r_3, g, g_2, g_2x\}$ ($[4] = 4$). $\xi = \eta\zeta$ with $g_3\eta = g_4$. We can also assume, by renumbering the vertices and/or reversing the cycle if necessary, that η has more than two letters and that $g_3\eta' \neq g_1v$.

If $g_3\varphi(\eta)$ is also different from g_1v then let $(g_1, \mu), (g_3\eta' v, \nu)$ be a $2-$path covering of $\mathcal{Q}_n^{top} - \{r_1, r_2\}$ that connects g_1 to $g_3\varphi(\eta)v$, and $g_3\eta' v$ to g_2xv ($[2, 2, 0, 2] = 4$). The desired $2-$path covering of $\mathcal{Q}_n - \mathcal{F}$ is $(g_1, \mu v(\eta^*)' v\nu vx)$, (g_3, ζ^R).

If $g_3\varphi(\eta) = g_1v$ then let $(g_3\eta' v, \mu)$ be a Hamiltonian path of $\mathcal{Q}_n^{top} - \{r_1, r_2, g_1\}$ that connects $g_3\eta' v$ to g_2xv ($[3, 1, 0, 1] = 4$). The desired $2-$path covering of $\mathcal{Q}_n - \mathcal{F}$ is $(g_1, v(\eta^*)' v\mu vx), (g_3, \zeta^R)$.

Subcase B6. All the green terminals are on the bottom plate.

Let (g_1, ξ) be a Hamiltonian cycle of $\mathcal{Q}_n^{bot} - \{r_3, g\}$. Such cycle exists for $[2] = 3$. Since the dimension of the plates are greater than or equal to 4 we can also assume that the distance from g_1 to g_2 along the cycle is at least 4 (Lemma 3.14). There are two essentially different distributions of the four green terminals along the cycle. In the first case $\xi = \eta\theta\zeta\kappa$ with $g_1\eta = g_2, g_2\theta = g_3, g_3\zeta = g_4$, where g_3, g_4 are to be renumbered, if necessary. In the second case $\xi = \eta\theta\zeta\kappa$ with $g_1\eta = g_3, g_3\theta = g_2, g_2\zeta = g_4$, where g_3, g_4 are to be renumbered, if necessary.

In the first case we proceed as follows. Let $(g_1\varphi(\eta)v, \mu), (g_1\eta' v, \nu)$ be a $2-$path covering of $\mathcal{Q}_n^{top} - \{r_1, r_2\}$ that connects $g_1\varphi(\eta)v$ to $g_1(\kappa^R)' v$ and $g_1\eta' v$ to $g_2\theta' v$. Then the desired $2-$path covering of $\mathcal{Q}_n - \mathcal{F}$ for this case is $(g_1, (\kappa^R)' v\mu^R v\eta'^* v\nu v(\theta')^R)$, (g_3, ζ).

In the second case we proceed as follows. Let $(g_1\eta' v, \mu), (g_3\theta' v, \nu)$ be a $2-$path covering of $\mathcal{Q}_n^{top} - \{r_1, r_2\}$ that connects $g_1\eta' v$ to $g_2\zeta' v$ and $g_3\theta' v$ to $g_4\kappa' v$. Such path covering exists since $[2, 2, 0, 2] = 4$. The desired $2-$path covering of $\mathcal{Q}_n - \mathcal{F}$ for this case is $(g_1, \eta' v\mu v(\zeta')^R)$, $(g_3, \theta' v\nu v(\kappa')^R)$. □

Lemma 5.7. $[2, 0, 0, 2] = 5$.

Proof. It follows from Lemma 5.4 that $[2,0,0,2] \leq [3,1,1,1]$ and since $[3,1,1,1] = 5$ (Lemma 5.6) we have $[2,0,0,2] \leq 5$. The following counterexample shows that $[2,0,0,2] \geq 5$.

Let $r = (0,1,1,0)$, $r_1 = (0,0,1,1)$, $r_2 = (0,1,0,1)$, $g = (1,1,0,1)$, $g_1 = (1,0,1,1)$, $g_2 = (1,1,1,0)$ be vertices in $\mathcal{Q}_4$. Then it is not difficult to verify that a $2-$path covering of $\mathcal{Q}_4 - \{r, g\}$ with path γ_1 connecting r_1 to r_2 and path γ_2 connecting g_1 to g_2 does not exist. □

Lemma 5.8. ($[5,1,0,1] = 5$) *Let $n \geq 5$ and $\mathcal{F} = \{r_1, r_2, r_3, g_1, g_2\}$ be a fault with three distinct red and two distinct green vertices. If $g_3, g_4 \in \mathcal{Q}_n - \mathcal{F}$ are two distinct green vertices then there exists a Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ that connects g_3 to g_4. The claim is not true if $n = 3$ or $n = 4$.*

Proof. It follows from Lemma 5.3 that if $k \geq 4$ and $k \in \mathcal{A}_{5,1,0,1}$ then k is in $\mathcal{A}_{4,0,1,0}$ and since $[4,0,1,0] = 5$ we have $[5,1,0,1] \geq 5$. We shall prove that $[5,1,0,1] = 5$. Let $n \geq 5$. Split $\mathcal{Q}_n$ into two plates in a way that two red vertices, say r_1 and r_2, are on the top plate and r_3 is on the bottom plate. We shall consider all essentially different cases depending on the distribution of the two green deleted vertices and the two green terminals.

Case A. The two green deleted vertices are on the top plate.

Subcase A1. g_3 and g_4 are on the top plate.

Use $[4] = 4$ to find a Hamiltonian cycle (g_3, ξ) of $\mathcal{Q}_n^{top} - \{r_1, r_2, g_1, g_2\}$. Let $\xi = \eta\theta$, with $g_3\eta = g_4$. Use $[1,1,0,1] = 2$ to find a Hamiltonian path $(g_3\eta' v, \zeta)$ of $\mathcal{Q}_n^{bot} - \{r_3\}$ that connects $g_3\eta' v$ to $g_3\xi' v$. The desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ for this case is $(g_3, \eta' v \zeta v (\theta')^R)$.

Subcase A2. g_3 is on the top plate and g_4 is on the bottom plate.

Use $[4] = 4$ to find a Hamiltonian cycle (g_3, ξ) of $\mathcal{Q}_n^{top} - \{r_1, r_2, g_1, g_2\}$. Either $g_3\varphi(\xi)$ or $g_3\xi'$ is not adjacent to g_4. Assume, without loss of generality, that $g_3\xi'$ is not adjacent to g_4. Use $[1,1,0,1] = 2$ to find a Hamiltonian path $(g_3\xi' v, \eta)$ of $\mathcal{Q}_n^{bot} - \{r_3\}$ that connects $g_3\xi' v$ to g_4. The desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ for this case is $(g_3, \xi' v \eta)$.

Subcase A3. g_3 and g_4 are on the bottom plate.

Use $[4] = 4$ to find a Hamiltonian cycle γ of $\mathcal{Q}_n^{top} - \{r_1, r_2, g_1, g_2\}$. Let a, b be two consecutive vertices along this cycle such that neither av nor bv is a deleted vertex or a terminal and let $\gamma = (a, \xi)$, with $a\xi' = b$. Use $[1,1,1,1] = 4$ to find a $2-$path covering $(av, \eta), (bv, \theta)$ of $\mathcal{Q}_n^{bot} - \{r_3\}$ that connects av to g_3 and bv to g_4. The desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ for this case is $(g_3, \eta^R v \xi' v \theta)$.

Case B. g_1 is on the top plate and g_2 is on the bottom plate.

Subcase B1. g_3 and g_4 are on the top plate.

Use $[3,1,0,1] = 4$ to find a Hamiltonian path (g_1, ξ) of $\mathcal{Q}_n^{top} - \{r_1, r_2, g_1\}$ that connects g_3 to g_4. Since $n \geq 5$ there exist words η, θ and a letter x such that $\xi = \eta x \theta$, and neither $g_3\eta v$ nor $g_3\eta x v$ is a deleted vertex. Use $[2,0,1,0] = 4$ to find a Hamiltonian path $(g_3\eta v, \zeta)$ of $\mathcal{Q}_n^{bot} - \{r_3, g_2\}$ that connects $g_3\eta v$ to $g_3\eta x v$. The desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ for this case is $(g_3, \eta v \zeta v \theta)$.

Subcase B2. g_3 is on the top plate and g_4 is on the bottom plate.

Let g_5 be a green vertex on the top plate such that neither g_5 nor g_5v is a deleted vertex or a terminal. Use $[3,1,0,1]=4$ to find a Hamiltonian path (g_3,ξ) of $\mathcal{Q}_n^{top}-\{r_1,r_2,g_1\}$ that connects g_3 to g_5. Use $[2,0,1,0]=4$ to find a Hamiltonian path (g_5v,η) of $\mathcal{Q}_n^{bot}-\{r_3,g_2\}$ that connects g_5v to g_4. The desired Hamiltonian path of $\mathcal{Q}_n-\mathcal{F}$ for this case is $(g_3,\xi v\eta)$.

Subcase B3. g_3 and g_4 are on the bottom plate.

Let g_5 and g_6 be any two green vertices on the top plate different from g_1 such that neither g_5v nor g_6v is a deleted vertex (clearly they cannot be terminal vertices). Use $[3,1,0,1]=4$ to find a Hamiltonian path (g_5,ξ) of $\mathcal{Q}_n^{top}-\{r_1,r_2,g_1\}$ that connects g_5 to g_6. Use $[2,0,2,0]=4$ to find a $2-$path covering (g_3,η), (g_4,θ) of $\mathcal{Q}_n^{bot}-\{r_3,g_2\}$ that connects g_3 to g_5v and g_4 to g_6v. The desired Hamiltonian path of $\mathcal{Q}_n-\mathcal{F}$ for this case is $(g_3,\eta v\xi v\theta^R)$.

Case C. The two green deleted vertices are on the bottom plate.

Subcase C1. g_3 and g_4 are on the top plate.

Let g_5 and g_6 be any two green vertices on the top plate different from g_3 and g_4 such that $g_5v\neq r_3$ and $g_6v\neq r_3$. Use $[2,0,2,0]=4$ to find a $2-$path covering (g_3,ξ), (g_4,η) of $\mathcal{Q}_n^{top}-\{r_1,r_2\}$ that connects g_3 to g_5 and g_4 to g_6. Use $[3,1,0,1]=4$ to find a Hamiltonian path (g_5v,ζ) of $\mathcal{Q}_n^{bot}-\{r_3,g_1,g_2\}$ that connects g_5v to g_6v. The desired Hamiltonian path of $\mathcal{Q}_n-\mathcal{F}$ for this case is $(g_3,\xi v\zeta v\eta^R)$.

Subcase C2. g_3 is on the top plate and g_4 is on the bottom plate.

Let r_4 be a red vertex on the bottom plate such that neither r_4 nor r_4v is a deleted vertex or a terminal. Use $[4]=4$ to find a Hamiltonian cycle (g_4,ξ) of $\mathcal{Q}_n^{bot}-\{r_3,r_4,g_1,g_2\}$. By replacing ξ with ξ^R, if necessary, we can assume that $g_4\xi'v\neq g_3$. Since the bottom plate is of dimension at least 4, there exists a letter y such that $g_5=r_4y$ is neither a terminal nor a deleted vertex. Let $\xi=\eta\theta$ with $g_4\eta=g_5$. Set $g_6=g_4\eta'v$ or $g_6=g_4\eta\varphi(\theta)v$, making sure that $g_6\neq g_3$. Use $[2,2,0,2]=4$ to find a $2-$path covering (g_3,μ), (g_6,ν) of $\mathcal{Q}_n^{top}-\{r_1,r_2\}$ that connects g_3 to $g_4\xi'v$ and g_6 to r_4v. The desired Hamiltonian path of $\mathcal{Q}_n-\mathcal{F}$ for this case is $(g_4,\eta'v\nu vy\theta'v\mu^R)$ if $g_6=g_4\eta'v$ or $(g_4,\eta yv\nu^Rv\theta'^*v\mu^R)$ if $g_6=g_4\eta\varphi(\theta)v$.

Subcase C3. g_3 and g_4 are on the bottom plate.

Let r_4 and r_5 be any two red vertices on the bottom plate that are not deleted vertices. Use $[3,1,0,1]=4$ to find a Hamiltonian path (r_4,ξ) of $\mathcal{Q}_n^{bot}-\{r_3,g_1,g_2\}$ that connects r_4 to r_5 and let $\xi=\eta\theta\mu$, with $r_4\eta=g_3$ and $r_4\eta\theta=g_4$, where g_3 and g_4 should be renumbered, if necessary. If the length of η is at least three then use $[2,2,0,2]=4$ to find a $2-$path covering $(r_4\eta'v,\nu),(g_3\theta'v,\zeta)$ of $\mathcal{Q}_n^{top}-\{r_1,r_2\}$ that connects $r_4\eta'v$ to r_5v and $g_3\theta'v$ to r_4v. The desired Hamiltonian path of $\mathcal{Q}_n-\mathcal{F}$ for this case is $(g_3,\theta'v\zeta v\eta'v\nu v\mu^R)$.

The case when the length of μ is at least three is equivalent to the case when the length of η is at least three.

If η and μ are both of length one then θ is of length greater than three. In this case use $[2,2,0,2]=4$ to produce a $2-$path covering $(r_4v,\nu),(g_3\varphi(\theta)v,\zeta)$

of $\mathcal{Q}_n^{top} - \{r_1, r_2\}$ that connects r_4v to $g_3\theta'v$ and $g_3\varphi(\theta)v$ to r_5v. The desired Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ for this case is $(g_3, \eta^R v\nu v(\theta'^*)^R v\zeta v\mu^R)$. □

Lemma 5.9. *($[3,3,0,3] \leq 6$) Let $n \geq 6$ and $\mathcal{F} = \{r_1, r_2, r_3\}$ be a fault in $\mathcal{Q}_n$ with three distinct red vertices. If g_1, g_2, g_3, g_4, g_5, g_6 are six distinct green vertices in $\mathcal{Q}_n - \mathcal{F}$ then there exists a $3-$path covering of $\mathcal{Q}_n - \mathcal{F}$ that connects g_1 to g_2, g_3 to g_4, and g_5 to g_6.*

Proof. Split $\mathcal{Q}_n$ into two plates with two red vertices, say r_1 and r_2, on the top plate, and r_3 on the bottom plate. We consider several cases that depend on the distribution of the green terminals on the plates.

Case 1. All the green terminals are on the top plate.

Without loss of generality we can assume that $g_6v \neq r_3$. Let x be a letter such that g_5x is not a deleted vertex. Let $(g_1, \xi), (g_3, \eta)$ be a $2-$path covering of $\mathcal{Q}_n^{top} - \{r_1, r_2, g_5, g_5x\}$ that connects g_1 to g_2 and g_3 to g_4. Such path covering exists since $[4, 2, 0, 2] = 5$. Without loss of generality we can assume that g_6 lies on the path from g_3 to g_4. Let $\eta = \theta\zeta$ with $g_3\theta = g_6$ and let $(g_5xv, \mu), (g_3\theta'v, \nu)$ be a $2-$path covering of $\mathcal{Q}_n^{bot} - \{r_3\}$ that connects g_5xv to g_6v and $g_3\theta'v$ to $g_6\varphi(\zeta)v$. Such path covering exists since $[1, 1, 1, 1] = 4$. The desired $3-$path covering of $\mathcal{Q}_n - \mathcal{F}$ for this case is $(g_1, \xi), (g_3, \theta'v\nu v\zeta^*), (g_5, xv\mu v)$.

Case 2. g_1, g_2, g_3, g_4, g_5 are on the top plate and g_6 is on the bottom plate.

Let x be a letter such that g_5x is not a deleted vertex and $g_5xv \neq g_6$. Let $(g_1, \xi), (g_3, \eta)$ be a $2-$path covering of $\mathcal{Q}_n^{top} - \{r_1, r_2, g_5, g_5x\}$ that connects g_1 to g_2 and g_3 to g_4. Such path covering exists since $[4, 2, 0, 2] = 5$. Let (g_5xv, μ) be a Hamiltonian path of $\mathcal{Q}_n^{bot} - \{r_3\}$ that connects g_5xv to g_6. Such path exists since $[1, 1, 0, 1] = 2$. The desired $3-$path covering of $\mathcal{Q}_n - \mathcal{F}$ for this case is $(g_1, \xi), (g_3, \eta), (g_5, xv\mu)$.

Case 3. $g_1, g_2, g_3, g_4,$ are on the top plate and g_5, g_6 are on the bottom plate.

Here we simply connect g_1 to g_2 and g_3 to g_4 by a $2-$path covering of $\mathcal{Q}_n^{top} - \{r_1, r_2\}$ and g_5 to g_6 by a Hamiltonian path of $\mathcal{Q}_n^{bot} - \{r_3\}$. That produces the desired $3-$path covering of $\mathcal{Q}_n - \mathcal{F}$ for this case.

Case 4. g_1, g_2, g_3, g_5 are on the top plate and g_4, g_6 are on the bottom plate.

Let x be a letter such that $g_3xv \neq g_4, g_6$, and let g be any green vertex on the top plate such that $gv \neq r_3$. Let $(g_1, \xi), (g_5, \eta)$ be a $2-$path covering of $\mathcal{Q}_n^{top} - \{r_1, r_2, g_3, g_3x\}$ that connects g_1 to g_2 and g_5 to g. Such path covering exists since $[4, 2, 0, 2] = 5$. Let $(g_3xv, \mu), (gv, \nu)$ be a $2-$path covering of $\mathcal{Q}_n^{bot} - \{r_3\}$ that connects g_3xv to g_4 and gv to g_6. Such path covering exists since $[1, 1, 1, 1] = 4$. The desired $3-$path covering of $\mathcal{Q}_n - \mathcal{F}$ for this case is $(g_1, \xi), (g_3, xv\mu), (g_5, \eta v\nu)$.

Case 5. $g_1, g_2, g_3,$ are on the top plate and g_4, g_5, g_6 are on the bottom plate.

Let g be a green vertex on the top plate such that $gv \neq r_3$. Let $(g_1, \xi), (g_3, \eta)$ be a $2-$path covering of $\mathcal{Q}_n^{top} - \{r_1, r_2\}$ that connects g_1 to g_2 and g_3 to g. Such path covering exists since $[2, 2, 0, 2] = 4$. Let $(gv, \mu), (g_5, \nu)$ be a $2-$path covering of $\mathcal{Q}_n^{bot} - \{r_3\}$ that connects gv to g_4 and g_5 to g_6. Such path covering exists since $[1,1,1,1]=4$. The desired $3-$path covering of $\mathcal{Q}_n-\mathcal{F}$ for this case is $(g_1,\xi),(g_3,\eta v\mu),(g_5,\nu)$.

Case 6. g_1,g_3,g_5 are on the top plate and g_2,g_4,g_6 are on the bottom plate.

Without loss of generality we can assume that $g_5v\neq r_3$. Since g_1 and g_3 together have at least eight neighbors in $\mathcal{Q}_n^{top}$ (Lemma 5.2) and there are only two deleted red vertices on the top plate and three green terminals on the bottom plate, we can also assume, renumbering g_1 and g_3, if necessary, that there is a letter x such that g_3xv is not a terminal and g_3x is not a deleted vertex. Finally, let y be a letter such that $g_2y\neq r_3$ and g_2yv is not a terminal. Let (g_1,η) be a Hamiltonian path of $\mathcal{Q}_n^{top}-\{r_1,r_2,g_3,g_3x,g_5\}$ that connects g_1 to g_2yv. Such path exists since $[5,1,0,1]=5$. Let $(g_3xv,\theta),(g_5v,\zeta)$ be a $2-$path covering of $\mathcal{Q}_n^{bot}-\{r_3,g_2,g_2y\}$ that connects g_3xv to g_4 and g_5v to g_6. Such path exists since $[3,1,1,1]=5$. The desired $3-$path covering of $\mathcal{Q}_n-\{r_1,r_2,r_3\}$ for this case is $(g_1,\eta vy),(g_3,xv\theta),(g_5,v\zeta)$.

Case 7. g_1,g_2 are on the top plate and g_3,g_4,g_5,g_6 are on the bottom plate.

Let x,y be letters such that neither g_5xv nor g_6yv is a terminal vertex. Let (g_1,ξ), (g_5xv,η) be a $2-$path covering of $\mathcal{Q}_n^{top}-\{r_1,r_2\}$ that connects g_1 to g_2 and g_5xv to g_6yv ($[2,2,0,2]=4$). Let (g_3,μ) be a Hamiltonian path of $\mathcal{Q}_n^{bot}-\{r_3,g_5,g_5x,g_6,g_6y\}$ that connects g_3 to g_4. Such path exists since $[5,1,0,1]=5$. The desired $3-$path covering of $\mathcal{Q}_n-\{r_1,r_2,r_3\}$ for this case is (g_1,ξ), (g_3,μ), $(g_5,xv\eta vy)$.

Case 8. g_1,g_3 are on the top plate and g_2,g_4,g_5,g_6 are on the bottom plate.

Let x be a letter such that $g_4x\neq r_3$ and g_4xv is not a terminal, and let g be any green vertex on the top plate such that $gv\neq r_3$. Let $(g_1,\xi),(g_3,\eta)$ be a $2-$path covering of $\mathcal{Q}_n^{top}-\{r_1,r_2\}$ that connects g_1 to g and g_3 to g_4xv ($[2,2,0,2]=4$). Let $(gv,\mu),(g_5,\nu)$ be a $2-$path covering of $\mathcal{Q}_n^{bot}-\{r_3,g_4,g_4x\}$ that connects gv to g_2 and g_5 to g_6. Such path covering exists since $[3,1,1,1]=5$. The desired $3-$path covering of $\mathcal{Q}_n-\{r_1,r_2,r_3\}$ for this case is $(g_1,\xi v\mu)$, $(g_3,\eta vx)$, (g_5,ν).

Case 9. g_1 is on the top plate and g_2,g_3,g_4,g_5,g_6 are on the bottom plate.

Assume that there exists a letter x such that $g_2x=g_1v\neq r_3$. Let (g_3,ξ), (g_5,η) be any $2-$path covering of $\mathcal{Q}_n^{bot}-\{r_3,g_2x\}$ that connects g_3 to g_4 and g_5 to g_6. Such path covering exists since $[2,2,0,2]=4$. Without loss of generality we can assume that g_2 lies on the path connecting g_3 to g_4. Let $\xi=\mu\nu$ with $g_3\mu=g_2$ and let $(g_3\mu'v,\zeta)$ be a Hamiltonian path of $\mathcal{Q}_n^{top}-\{r_1,r_2,g_1\}$ that connects $g_3\mu'v$ to $g_2\varphi(\nu)v$ ($[3,1,0,1]=4$). The desired $3-$path covering of $\mathcal{Q}_n-\{r_1,r_2,r_3\}$ for this case is $(g_1,vx),(g_3,\mu'v\zeta v\nu^*),(g_5,\eta)$.

If $g_1v=r_3$ or if the distance from g_1 to g_2 is greater than 2 we let x be any letter such that $g_2x\neq r_3$. Let $(g_3,\xi),(g_5,\eta)$ be any $2-$path covering of $\mathcal{Q}_n^{bot}-\{r_3,g_2x\}$ that connects g_3 to g_4 and g_5 to g_6 ($[2,2,0,2]=4$). Without loss of generality we can assume that g_2 lies on the path connecting g_3 to g_4. Let $\xi=\mu\nu$ with $g_3\mu=g_2$ and let $(g_1,\theta),(g_3\mu'v,\zeta)$ be a $2-$path covering of $\mathcal{Q}_n^{top}-\{r_1,r_2\}$ that connects g_1 to g_2xv and $g_3\mu'v$ to $g_2\varphi(\nu)v$

($[2,2,0,2]=4$). The desired $3-$path covering of $\mathcal{Q}_n-\{r_1,r_2,r_3\}$ for this case is $(g_1,\theta v x),(g_3,\mu' v\zeta v\nu^*),(g_5,\eta)$.

Case 10. All the green terminals are on the bottom plate.

Let x and y be any letters different from v. Let (g_1,ξ) be a Hamiltonian path of $Q_n^{bot}-\{r_3,g_5,g_5x,g_6,g_6y\}$ that connects g_1 to g_2. Such path exists since $[5,1,0,1]=5$. We can assume that $\xi=\eta\theta\zeta$ with $g_3=g_1\eta, g_4=g_3\theta$, by renumbering g_3 and g_4, if necessary. Let $(g_5xv,\mu),(g_1,\eta' v,\nu)$ be a $2-$path covering of $\mathcal{Q}_n^{top}-\{r_1,r_2\}$ that connects g_5xv to g_6yv and $g_1\eta' v$ to $g_4\varphi(\zeta)v$ ($[2,2,0,2]=4$). The desired $3-$path covering of $\mathcal{Q}_n-\{r_1,r_2,r_3\}$ for this case is $(g_1,\eta' v\nu v\zeta^*)$, (g_3,θ), $(g_5,xv\mu vy)$. □

The following corollary follows directly from Lemma 4.7, Lemma 5.9, and Lemma 5.3.

Corollary 5.10. $5=[0,0,3,0]\leq[1,1,2,1]\leq[2,2,1,2]\leq 6$.

Corollary 5.11. $[0,0,1,2]\leq 6$, $5\leq[1,1,0,3]\leq 6$, *and* $[5,1,1,1]\geq 5$.[1]

Proof. The upper bounds of the first two inequalities follow directly from Corollary 5.10 and Lemma 5.4. The last inequality follows from Lemma 5.5 and the fact that $[4,2,0,2]=5$ (Lemma 5.6). The following counterexample shows that $[1,1,0,3]\geq 5$.

Let $n=4$ and $r=(0,1,1,0)$. Let also $r_1=(0,0,1,1)$, $r_2=(0,1,0,1)$, $g_1=(1,0,1,1)$, $g_2=(1,1,1,0)$, $g_3=(1,1,0,1)$, and $g_4=(1,0,0,0)$ be vertices in $\mathcal{Q}_4-\{r\}$. Then one can directly verify that a $3-$path covering of $\mathcal{Q}_4-\{r\}$ with paths connecting r_1 to r_2, g_1 to g_2, and g_3 to g_4 does not exist. □

Lemma 5.12. ($[8]=6$) *Let $n\geq 6$ and $\mathcal{F}$ be any neutral fault of mass eight in $\mathcal{Q}_n$. Then $\mathcal{Q}_n-\mathcal{F}$ is Hamiltonian.*

Proof. Let $n\geq 6$. We split $\mathcal{Q}_n$ into two plates so that each plate has at least one red deleted vertex. There are two general cases: *Case A* – there are two red deleted vertices on each plate and *Case B* – there are three red deleted vertices on the top plate and one red deleted vertex on the bottom plate. Within each general case there are subcases that depend on the distribution of the green deleted vertices on the plates.

Let the fault be $\mathcal{F}=\{r_1,r_2,r_3,r_4,g_1,g_2,g_3,g_4\}$ with the r_i red and the g_i green.

Case A. r_1,r_2 are on the top plate and r_3,r_4 are on the bottom plate.

Subcase A1. All the green deleted vertices are on the top plate.

Let (g_1,ξ), (g_2,η) be a $2-$path covering of $\mathcal{Q}_n^{top}-\{r_1,r_2\}$ that connects g_1 to g_3 and g_2 to g_4. Such path covering exists since $[2,2,0,2]=4$. Let $(g_1\xi' v,\mu)$, $(g_1\varphi(\xi)v,\nu)$ be a $2-$path covering of $\mathcal{Q}_n^{bot}-\{r_3,r_4\}$ that connects $g_1\xi' v$ to

[1]While this paper was under review the authors were able to prove that $[0,0,1,2]=4$ ([4]), $[1,1,0,3]=5$, $[1,1,2,1]=5$ ([6]), $[4,0,2,0]=5$ and $[7,1,0,1]=6$ ([5]).

$g_2\eta' v$ and $g_1\varphi(\xi)v$ to $g_2\varphi(\eta)v$. Such path covering exists since $[2,2,0,2]=4$. The desired Hamiltonian cycle for this case is $(g_1\varphi(\xi), \xi'^* v\mu v(\eta'^*)^R v\nu^R v)$.

Subcase A2. g_1, g_2, g_3 are on the top plate and g_4 is on the bottom plate.

Let r_5, r_6 be any two non-deleted red vertices on the top plate such that neither $r_5 v$ nor $r_6 v$ is a deleted vertex. Let (r_5, ξ) be a Hamiltonian path of $\mathcal{Q}_n^{top} - \{r_1, r_2, g_1, g_2, g_3\}$ that connects r_5 to r_6. Such path exists since $[5,1,0,1]=5$. Let $(r_6 v, \eta)$ be a Hamiltonian path of $\mathcal{Q}_n^{bot} - \{r_3, r_4, g_4\}$ that connects $r_6 v$ to $r_5 v$. Such path exists since $[3,1,0,1]=4$. The desired Hamiltonian cycle for this case is $(r_5, \xi v\eta v)$.

Subcase A3. g_1, g_2 are on the top plate and g_3, g_4 are on the bottom plate.

Let r, g be a red and a green non-deleted vertices on the top plate such that neither rv nor gv is a deleted vertex. Let (r, ξ) be a Hamiltonian path of $\mathcal{Q}_n^{top} - \{r_1, r_2, g_1, g_2\}$ that connects r to g. Such path exists since $[4,0,1,0]=5$. Let (gv, η) be a Hamiltonian path of $\mathcal{Q}_n^{bot} - \{r_3, r_4, g_3, g_4\}$ that connects gv to rv. Such path exists since The desired Hamiltonian cycle for this case is $(r, \xi v\eta v)$.

Case B. r_1, r_2, r_3 are on the top plate and r_4 is on the bottom plate.

Subcase B1. All the green deleted vertices are on the top plate.

Let (g_1, ξ) be a Hamiltonian path for $\mathcal{Q}_n^{top} - \{r_1, r_2, r_3, g_3, g_4\}$ that connects g_1 to g_2. Such path exists since $[5,1,0,1]=5$. Let $(g_1\xi' v, \eta)$ be a Hamiltonian path of $\mathcal{Q}_n^{bot} - \{r_4\}$ that connects $g_1\xi' v$ to $g_1\varphi(\xi)v$. Such path exists since $[1,1,0,1]=2$. The desired Hamiltonian cycle for this case is $(g_1\varphi(\xi), \xi'^* v\eta v)$.

Subcase B2. g_1, g_2, g_3 are on the top plate and g_4 is on the bottom plate.

Let γ be any Hamiltonian cycle of $\mathcal{Q}_n^{top} - \{r_1, r_2, r_3, g_1, g_2, g_3\}$. Such cycle exists since $[6]=5$. We can find a vertex g on this cycle such that $\gamma = (g, \xi)$ with neither gv nor $g\xi' v$ being a deleted vertex. Let $(g\xi' v, \eta)$ be a Hamiltonian path of $\mathcal{Q}_n^{bot} - \{r_4, g_4\}$ that connects $g\xi' v$ to gv. The desired Hamiltonian cycle for this case is $(g, \xi' v\eta v)$.

Subcase B3. g_1 is on the top plate and g_2, g_3, g_4 are on the bottom plate.

Let g_5, g_6, g_7, g_8 be any green non-deleted vertices on the top plate such that none of $g_5 v, g_6 v, g_7 v, g_8 v$ is a deleted vertex. Let $(g_5, \xi), (g_7, \eta)$ be a 2−path covering of $\mathcal{Q}_n^{top} - \{r_1, r_2, r_3, g_1\}$ that connects g_5 to g_6 and g_7 to g_8. Such path covering exists since $[4,2,0,2]=5$. Let $(g_6 v, \mu), (g_8, \nu)$ be a 2−path covering of $\mathcal{Q}_n^{bot} - \{r_4, g_2, g_3, g_4\}$ that connects $g_6 v$ to $g_7 v$ and $g_8 v$ to $g_5 v$. Such path covering exists since $[4,2,0,2]=5$. The desired Hamiltonian cycle for this case is $(g_5, \xi v\mu v\eta v\nu v)$.

Subcase B4. g_1, g_2 are on the top plate and g_3, g_4 are on the bottom plate.

This case is equivalent to *Subcase A2.*

Subcase B5. All the green deleted vertices are on the bottom plate.

This case can be avoided if $n=6$. Indeed, if the four deleted red vertices are contained in a three dimensional subcube of $\mathcal{Q}_6$ then we can split $\mathcal{Q}_6$ into two plates with 2 deleted red vertices on each plate. If the four deleted red verctices are not contained in any three dimensional subcube of $\mathcal{Q}_6$ then there are at least 4 coordinates that split the red vertices. At least one of these coordinates must split the green deleted vertices as well, for otherwise the 4 green deleted vertices would have to be contained in a two dimensional subcube which is impossible. Therefore, for this case we assume that $n \geq 7$.

Let (g_1, ξ) be a Hamiltonian path of $\mathcal{Q}_n^{bot} - \{r_4\}$ that connects g_1 to g_4. Such path exists since $[1, 1, 0, 1] = 2$. It follows from Lemma 4.4, renumbering g_2 and g_3, if necessary, that $\xi = \eta\theta\zeta$ with $g_1\eta = g_2$, $g_2\theta = g_3$, $g_3\zeta = g_4$ and the words η, θ, and ζ each of length at least 4. Let $(g_1\varphi(\eta)v, \kappa)$, $(g_3\varphi(\zeta)v, \mu)$, and $(g_2\varphi(\theta)v, \nu)$ be a 3−path covering of $\mathcal{Q}_n^{top} - \{r_1, r_2, r_3\}$ that connects $g_1\varphi(\eta)v$ to $g_1\xi'v$, $g_3\varphi(\zeta)v$ to $g_2\theta'v$, and $g_2\varphi(\theta)v$ to $g_1\eta'v$. The existence of such path covering follows from Lemma 5.9. The desired Hamiltonian cycle for this case is $(g_1\varphi(\eta), v\kappa v(\zeta'^*)^R v\mu v(\theta'^*)^R v\nu v(\eta'^*)^R$. □

6. Concluding Remarks and Conjectures

We have found several values of $[M, C, N, O]$ when the parameters involved are relatively small. Unfortunately, as the parameters increase the number of cases to be considered in the proofs becomes extremely large. We hope that further analysis and improvement of our proofs will lead to substantial simplifications. Our results support the following conjectures:

Conjecture 6.1. *(Locke [16]) Let $k \geq 0$. Then $[2k] = k + 2$.*

We have already discussed that $[2k] \geq k + 2$. And after this paper we know that the conjecture is true for $0 \leq k \leq 4$. The proof of this conjecture for $k \geq 5$, which depends on the proof of Conjecture 6.2, is contained in [3].

Conjecture 6.2. *Let $k \geq 1$. Then $[2k + 1, 1, 0, 1] = k + 3$.*

In this article we have proved that this conjecture is true for $k = 1$ and $k = 2$ and the proof for the case $k = 3$ is contained in [5]. The proof of this conjecture for $k \geq 4$, which depends on the proof of Conjecture 6.1, is contained in [7]. Here we can show that $[2k + 1, 1, 0, 1] \geq k + 3$. Indeed, let r be any red vertex in $\mathcal{Q}_{k+2}$ and $\mathcal{F}$ be a fault of mass $2k + 1$ that contains any $k + 1$ red vertices different from r and all the green vertices adjacent to r except two vertices g_1 and g_2. Then, obviously, the only path in $\mathcal{Q}_{k+2} - \mathcal{F}$ that connects g_1 to g_2 and visits r is of length 3 and cannot be a Hamiltonian path of $\mathcal{Q}_{k+2} - \mathcal{F}$ if $k \geq 1$.

The following conjecture is a direct corollary of Conjecture 6.2.

Conjecture 6.3. *Let $k \geq 1$. Then $[2k, 0, 1, 0] = k + 3$.*

In this article we have proved this conjecture for $k = 1$ and $k = 2$. Let us prove that $[2k, 0, 1, 0] \geq k + 3$.

Let $x_1, x_2, \ldots, x_{k+2}$ be the standard generators of $\mathbf{Z}_2^{k+2}$. We select any red vertex r in $\mathcal{Q}_{k+2}$ and set

$$\mathcal{F} = \{rx_1, rx_2, \ldots, rx_k, rx_{k+2}x_1, rx_{k+2}x_2, \ldots, rx_{k+2}x_k\}.$$

Then the only path that connects rx_{k+1} to $rx_{k+1}x_{k+2}$ and visits r is of length 3 and cannot be a Hamiltonian path of $\mathcal{Q}_{k+2} - \mathcal{F}$ if $k \geq 1$.

Conjecture 6.4. *Let $k \geq 0$. Then $[2k+1, 1, 1, 1] = k + 4$.*

In this article we have proved this conjecture for $k = 0, 1$. Let us prove that $[2k+1, 1, 1, 1] \geq k + 4$.

Let $\{x_1, x_2, \ldots, x_{k+3}\}$ be the standard generators of $\mathbf{Z}_2^{k+3}$. We select any red vertex r in $\mathcal{Q}_{k+3}$ and set

$$\mathcal{F} = \{rx_1, rx_2, \ldots, rx_k, rx_{k+3}x_1, rx_{k+3}x_2, \ldots, rx_{k+3}x_{k+1}\}.$$

Then there does not exist a 2−path covering of $\mathcal{Q}_{k+3} - \mathcal{F}$ that connects rx_{k+1} to rx_{k+2} and $rx_{k+2}x_{k+3}$ to any green vertex $g \notin \mathcal{F}$ for r and rx_{k+3} are blocked between all deleted and terminal vertices.

Even though our main focus in this article is the production of path coverings with prescribed ends for the hypercube with or without deleted vertices, we occasionally have considered the more general problem of prescribing ends and edges. The following conjecture is related to this problem.

Conjecture 6.5. *Let $k \geq 0$ and $n \geq k + 4$. Let also $\mathcal{F}$ be any fault in $\mathcal{Q}_n$ with $k + 1$ red vertices and k green vertices, g_1 and g_2 be two green vertices in $\mathcal{Q}_n - \mathcal{F}$, and $e = \{a, b\}$ be any edge different from $\{g_1, g_2\}$ and not incident to any of the vertices of $\mathcal{F}$. Then there exists a Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ that connects g_1 to g_2 and passes through the edge e.*

In this article we have proved this conjecture for $k = 0$. To see that $n \geq k+4$, assume that $n = k + 3$, and let r and $\mathcal{F}$ be selected as in the discussion of Conjecture 6.4. Let $g_1 = rx_{k+1}, g_2 = rx_{k+2}$, and $e = \{g_2, rx_{k+3}x_{k+2}\}$. Then the only path in $\mathcal{Q}_n - \mathcal{F}$ that connects g_1 to g_2, passes through e, and visits rx_{k+3} is $g_2, rx_{k+3}x_{k+2}, rx_{k+3}, r, g_1$ which obviously is not a Hamiltonian path of $\mathcal{Q}_{k+3} - \mathcal{F}$.

Finally, we point out that in [2] we use results from this article to obtain the following generalization of a theorem of Fu [12]:

Theorem 6.6. *([2]) Let f and n be integers with $n \geq 5$ and $0 \leq f \leq 3n-7$. Then for any set of vertices $\mathcal{F}$ of cardinality f in $\mathcal{Q}_n$ there exists a cycle in $\mathcal{Q}_n - \mathcal{F}$ of length at least $2^n - 2f$.*

Appendix A. 2−path coverings of $\mathcal{Q}_4$

When a neutral pair is deleted from $\mathcal{Q}_4$ one can still freely prescribe the ends for a 2−path covering of the resulting graph. In spite the fact that the dimension is so low we find it difficult to verify this statement by inspection. Therefore, we provide a proof below for the benefit of the reader.

Lemma A.1. *Let $\mathcal{F} = \{r, g\}$ be a neutral fault in $\mathcal{Q}_4$, and r_1, r_2, g_1, g_2 be two red and two green vertices in $\mathcal{Q}_4 - \mathcal{F}$. Then there exists a 2−path covering of $\mathcal{Q}_4 - \mathcal{F}$ with one path connecting r_1 to g_1 and the other connecting r_2 to g_2.*

Proof. The deleted vertices r and g have opposite parity and belong to $\mathcal{Q}_4$. Therefore we can split $\mathcal{Q}_4$ in such way that both vertices belong to the same plate, say $\mathcal{Q}_4^{top}$. We consider all essentially different cases that depend on the distribution of the vertices r_1, r_2, g_1, g_2 between the plates.

Case 1. $r_1, r_2, g_1, g_2 \in \mathcal{Q}_4^{top}$.

Subcase 1(a). Let $\{r_1, g_1\}, \{r_2, g_2\} \in \mathcal{B}_{\{r,g\}}$. Then there exists a one-letter word x such that $(r_1, x), (g_1, x)$ is a 2−path covering of $\mathcal{Q}_4^{top} - \{r, g, r_2, g_2\}$. Let $(r_1 xv, \mu), (r_2 v, \nu)$ be a 2−path covering of $\mathcal{Q}_4^{bot}$ that connects $r_1 xv$ to $g_1 xv$, and $r_2 v$ to $g_2 v$. Such path covering exists since $[0, 0, 2, 0] = 2$. The desired 2−path covering of $\mathcal{Q}_4 - \{r, g\}$ is $(r_1, xv\mu vx)$, $(r_2, v\nu v)$.

Subcase 1(b). If either $\{r_1, g_1\}$ or $\{r_2, g_2\}$ is not in $\mathcal{B}_{\{r,g\}}$ we can assume without loss of generality that $\{r_1, g_1\} \notin \mathcal{B}_{\{r,g\}}$. Then, according to Lemma 3.5(1), there exists a Hamiltonian path (r_1, ξ) of $\mathcal{Q}_4^{top} - \{r, g\}$ that connects r_1 to g_1. Let $\xi = \eta\theta\zeta$ with $(r_1\eta, r_1\eta\theta)$ equals (r_2, g_2) or (g_2, r_2). Let also $(r_1\eta' v, \mu)$ be a Hamiltonian path of $\mathcal{Q}_4^{bot}$ that connects $r_1\eta' v$ to $g_1(\zeta^R)' v$. The desired 2−path covering of $\mathcal{Q}_4 - \{r, g\}$ is $(r_1, \eta' v\mu v\zeta^*), (r_1\eta, \theta)$.

Case 2. r_1, r_2, g_1 are on the top plate and g_2 is on the bottom plate.

Subcase 2(a). If $\{g_1, r_2\} \notin \mathcal{B}_{\{r,g\}}$ then, according to Lemma 3.5(1), there exists a Hamiltonian path (g_1, ω) of $\mathcal{Q}_4^{top} - \mathcal{F}$ that connects g_1 to r_2. Let $\omega = \xi\eta$ with $g_1\xi = r_1$ and let $(r_1\varphi(\eta)v, \theta)$ be a Hamiltonian path of Q_4^{bot} that connects $r_1\varphi(\eta)v$ to g_2. The desired 2−path covering of $\mathcal{Q}_4 - \{r, g\}$ is $(g_1, \xi), (g_2, \theta^R v\eta^*)$.

Subcase 2(b). If $\{g_1, r_2\} \in \mathcal{B}_{\{r,g\}}$ then $\{g_1, r_1\} \notin \mathcal{B}_{\{r,g\}}$ and there exists a Hamiltonian path (g_1, ω) of $\mathcal{Q}_4 - \mathcal{F}$ that connects g_1 to r_1. Let $\omega = \xi\eta$ with $g_1\xi = r_2$. We have to consider two sub-subcases:

(i) $g_2 v = r_1$ or $g_2 v = r_2$.

We observe that the lengths of ξ and η are 1 and 4 or 3 and 2.

If ξ is the longer word, then we use $[0, 0, 2, 0] = 2$ to produce a 2−path covering $(g_1\xi' v, \mu), (g_1\xi'' v, \nu)$ of $\mathcal{Q}_4^{bot}$ that connects $g_1\xi' v$ to g_2 and $g_1\xi'' v$ to $r_2\varphi(\eta)v$. The desired 2−path covering of $\mathcal{Q}_4 - \{r, g\}$ is $(g_1, \xi'' v\nu v\eta^*), (g_2, \mu^R v\varphi(\xi^R))$.

If η is the longer word, then we use $[0, 0, 2, 0] = 2$ to produce a 2−path covering $(g_1\xi' v, \mu), (r_2\varphi(\eta)v, \nu)$ of $\mathcal{Q}_4^{bot}$ that connects $g_1\xi' v$ to $r_1(\eta^R)'' v$ and $r_2\varphi(\eta)v$ to g_2. The desired 2−path covering of $\mathcal{Q}_4 - \{r, g\}$ is $(g_1, \xi' v\mu v\eta^{**}), (r_2, \varphi(\eta)v\nu)$.

(ii) $g_2 v$ is an interior vertex of the path (g_1, ω).

If $\xi = \theta\zeta$ with $g_1\theta = g_2 v$ then we use $[1, 1, 0, 1] = 2$ to produce a Hamiltonian path $(g_1\theta' v, \mu)$ of $\mathcal{Q}_4^{top} - \{g_2\}$ that connects $g_1\theta' v$ to $r_2\varphi(\eta)v$. The desired 2−path covering of $\mathcal{Q}_4 - \{r, g\}$ is $(g_1, \theta' v\mu v\eta^*), (r_2, \zeta^R v)$.

If $\eta = \theta\zeta$ with $r_2\theta = g_2 v$ then we use $[1, 1, 0, 1] = 2$ to produce a Hamiltonian path $(g_1\xi' v, \mu)$ of $\mathcal{Q}_4^{top} - \{g_2\}$ that connects $g_1\xi' v$ to $r_1(\zeta^R)' v$. The desired 2−path covering of $\mathcal{Q}_4 - \{r, g\}$ is $(g_1, \xi' v\mu v\zeta^*), (r_2, \theta v)$.

Case 3. $r_1, r_2 \in \mathcal{Q}_4^{top}$ and $g_1, g_2 \in \mathcal{Q}_4^{bot}$.

Find a Hamiltonian path of $\mathcal{Q}_4^{top} - \{g\}$ that connects r_1 to r_2. The vertex r belongs to that path. Cut that path just before r and right after r and connect these two vertices with bridges to the bottom plate. Let r_3 and r_4 be the ends of these bridges that belong to the bottom plate. Then use $[0, 0, 2, 0] = 2$ to find a 2−path covering of the bottom plate that connects r_3 and r_4 to the appropriate vertices g_1 and g_2.

Case 4. $r_1, g_1 \in \mathcal{Q}_4^{top}$ and $r_2, g_2 \in \mathcal{Q}_4^{bot}$.

Consider $\mathcal{Q}_4^{top}$. It is not difficult to verify that either there is a Hamiltonian path for $\mathcal{Q}_4^{top} - \{r, g\}$ that connects r_1 to g_1 or there is a path with length 3 connecting r_1 to g_1 such that exactly one edge remains not covered. In the first case use $[0, 0, 1, 0] = 1$ to find a Hamiltonian path for $\mathcal{Q}_4^{bot}$ connecting r_2 to g_2. In the second case denote by r_3 and g_3 the vertices in the bottom plate that are neighbors of the vertices in $\mathcal{Q}_4^{top}$ that are not covered. Use Corollary 3.2 to find a Hamiltonian path for $\mathcal{Q}_4^{bot}$ that connects r_2 to g_2 and passes trough the edge $\{r_3, g_3\}$. Cut that path at that edge and using two bridges connect both pieces to the non-covered edge from the top plate.

Case 5. $r_1, g_2 \in \mathcal{Q}_4^{top}$ and $r_2, g_1 \in \mathcal{Q}_4^{bot}$.

We consider two subcases:

Subcase 5(a). Assume that $\{r_1, g_2\} \notin \mathcal{B}_{\{r,g\}}$. Then there is a Hamiltonian path (r_1, ξ) of $\mathcal{Q}_4^{top} - \{r, g\}$ that connects r_1 to g_2. There are three sub-subcases that depend on whether or not r_2 or g_1 are adjacent to vertices inside of the path (r_1, ξ).

(i) Assume that $\xi = \eta\theta$ with $r_1\eta v = g_1$. Let $(r_1\eta\varphi(\theta)v, \mu)$ be a Hamiltonian path of $\mathcal{Q}_4^{bot} - \{g_1\}$ that connects $r_1\eta\varphi(\theta)v$ to r_2. The desired 2−path covering of $\mathcal{Q}_4 - \mathcal{F}$ for this case is $(r_1, \eta v), (r_2, \mu^R v\theta^*)$.

(ii) Assume that $\xi = \eta\theta$ with $g_2\theta^R v = r_2$. Let $(r_1\eta' v, \mu)$ be a Hamiltonian path of $\mathcal{Q}_4^{bot} - \{r_2\}$ that connects $r_1\eta' v$ to g_1. The desired 2−path covering of $\mathcal{Q}_4 - \mathcal{F}$ for this case is $(r_1, \eta' v\mu), (r_2, v\theta)$.

(iii) Finally, let neither r_2 nor g_1 be adjacent to a vertex in the path (r_1, ξ). Let $\xi = xy\eta$ for some letters x, y, and a word η. Then there is a 2−path covering $(r_1xv, \mu), (r_1xyv, \nu)$ of $\mathcal{Q}_4^{bot}$ that connects r_1xv to g_1 and r_1xyv to r_2. The desired 2−path covering of $\mathcal{Q}_4 - \mathcal{F}$ for this case is $(r_1, xv\mu), (r_2, \nu^R v\eta)$.

Subcase 5(b). Let $\{r_1, g_2\} \in \mathcal{B}_{\{r,g\}}$. Then, according to Lemma 3.5, there exist two distinct 2−path coverings of $\mathcal{Q}_4^{top} - \{r, g\}$ with paths of length 2, one starting at r_1 and the other starting at g_2. We can choose a 2−path covering of $\mathcal{Q}_4^{top} - \{r, g\}$ to be $(r_1, \xi), (g_2, \eta)$, with $r_1\xi v \neq g_1$ or $g_2\eta v \neq r_2$. There are three sub-subcases:

(i) Let $r_1\xi v \neq g_1$ and $g_2\eta v \neq r_2$. Let $(r_1\xi v, \mu), (g_2\eta v, \nu)$ be a 2−path covering of $\mathcal{Q}_4^{bot}$ that connects $r_1\xi v$ to g_1 and $g_2\eta v$ to r_2. The desired 2−path covering of $\mathcal{Q}_4 - \mathcal{F}$ for this case is $(r_1, \xi v\mu), (g_2, \eta v\nu)$.

(ii) Let $r_1\xi v \neq g_1$ and $g_2\eta v = r_2$. Let $(r_1\xi v, \mu)$ be a Hamiltonian path of $\mathcal{Q}_4^{bot} - \{r_2\}$ that connects $r_1\xi v$ to g_1. The desired $2-$path covering of $\mathcal{Q}_4 - \mathcal{F}$ for this case is $(r_1, \xi v\mu), (g_2, \eta v)$.

(iii) Let $r_1\xi v = g_1$ and $g_2\eta v \neq r_2$. This case is completely symmetrical to case (ii).

Case 6. $r_1 \in \mathcal{Q}_4^{top}$ and $r_2, g_1, g_2 \in \mathcal{Q}_4^{bot}$.

Use Lemma 3.4 to find a Hamiltonian path of $\mathcal{Q}_4^{top} - \{g\}$ that connects r to r_1 and such that the vertex g_3 which is next to r in this path is not adjacent to r_2. Let the second end of the bridge that begins at g_3 be r_3. Use $[0, 0, 2, 0] = 2$ to find a $2-$path covering of the bottom plate that connects r_3 to g_1 and r_2 to g_2.

Case 7. $r_1, r_2, g_1, g_2 \in \mathcal{Q}_4^{bot}$.

Use $[0, 0, 2, 0] = 2$ to find a $2-$path covering of Q_4^{bot} that connects r_1 to g_1 and r_2 to g_2. Then find an edge that belongs to one of the two paths whose neighbors r_3 and g_3 in Q_4^{top} are not deleted vertices and also $\{r_3, g_3\} \notin \mathcal{B}_{\{r,g\}}$. Cut that path at that edge and use Lemma 3.5 to find a Hamiltonian path for $Q_4^{top} - \{r, g\}$ that connects r_3 to g_3. □

Appendix B

The following table summarizes some of the results obtained in this paper. The rows represent admissible combinations of M and C and the columns contain all the values of N and O such that $N + O \leq 3$. Each star in the table represents an impossible case. The missing entries in the table correspond to values of $[M, C, N, O]$ that we do not know yet. The inequalities in the table represent an upper or lower bound of the corresponding entry. Finally, the entries with an asterisk are results that were obtained after this paper was submitted for publication and therefore their proofs are not contained in this paper.

$MC\backslash NO$	01	10	20	11	02	30	21	12	03
00	$\star$	1	2	$\star$	4	5	$\star$	4^*	$\star$
11	2	$\star$	$\star$	4	$\star$	$\star$	5^*	$\star$	5^*
20	$\star$	4	4	$\star$	5		$\star$		$\star$
22	$\star$	$\star$	$\star$	$\star$	4	$\star$	$\star$	≤ 6	$\star$
31	4	$\star$	$\star$	5	$\star$	$\star$		$\star$	
33	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	≤ 6
40	$\star$	5	5^*	$\star$			$\star$		$\star$
42	$\star$	$\star$	$\star$	$\star$	5	$\star$	$\star$		$\star$
44	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
51	5	$\star$	$\star$	≥ 5	$\star$	$\star$		$\star$	

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