PATH COVERINGS WITH PRESCRIBED ENDS OF THE $n-$DIMENSIONAL BINARY HYPERCUBE

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ABSTRACT. Let $\mathcal{Q}_n$ be the $n-$dimensional binary hypercube, u_1, u_2 and u_3 be distinct even vertices of $\mathcal{Q}_n$ and v_1, v_2 and v_3 be distinct odd vertices of $\mathcal{Q}_n$. We prove that if $n \geq 4$, then there exist three paths in $\mathcal{Q}_n$, one joining u_1 and v_1, one joining u_2 and u_3 and one joining v_2 and v_3, such that every vertex of $\mathcal{Q}_n$ belongs to exactly one of the paths.

1. INTRODUCTION

A *path covering* of a graph $G = (V, E)$ is a set of vertex-disjoint simple paths in G with the property that every vertex of G is contained in one of the paths. A $k-$path covering of G is a path covering consisting of k paths. We are interested in $k-$path coverings of induced subgraphs of the $n-$dimensional binary hypercube $\mathcal{Q}_n$, where $k \geq 1$. An induced subgraph of $\mathcal{Q}_n$ is also known as a *faulty hypercube*.

The vertices of $\mathcal{Q}_n$ are binary vectors of length n. A vertex is *odd* if it contains an odd number of 1s and *even* if it contains an even number of 1s. In order to make our results as general as possible, the vertices of one parity will be called *green* and those of the other parity will be called *red*.

One of the most basic theorems on path coverings of $\mathcal{Q}_n$ is Havel's Lemma [H], which states that given any two vertices of opposite parity in $\mathcal{Q}_n$, with $n \geq 1$, there exists a Hamiltonian path with these two vertices as endpoints. This lemma was generalized by Dvořák:

Lemma 1.1 (Dvořák [D])**.** *Let $n \geq 2$, $g_1, g_2 \in V(\mathcal{Q}_n)$ be distinct green vertices and $r_1, r_2 \in V(\mathcal{Q}_n)$ be distinct red vertices. Then there exist two simple paths, one joining g_1 and r_1 and the other joining g_2 and r_2, such that each vertex of $\mathcal{Q}_n$ is contained in exactly one of the paths.*

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Castañeda and Gotchev continued the study of path coverings of induced subgraphs of $\mathcal{Q}_n$ in [CG]. In that article, they considered the following problem:

Problem 1.2. *Let M, C, N and O be nonnegative integers and $\mathcal{F}$ be a subset of $V(\mathcal{Q}_n)$ such that the cardinality of $\mathcal{F}$ is M and the difference between the number of odd vertices in $\mathcal{F}$ and the number of even vertices in $\mathcal{F}$ is C. Let also $a_1, \ldots, a_{N+O}$ and $b_1, \ldots, b_{N+O}$ be distinct vertices in $V(\mathcal{Q}_n) \setminus \mathcal{F}$, such that there are N pairs a_i, b_i of opposite parities and O pairs a_i, b_i of matching parities. Find the smallest m such that if $n \geq m$ then, regardless of the choice of $\mathcal{F}$ and $a_1, \ldots, a_{N+O}, b_1, \ldots, b_{N+O}$, there exists an $(N+O)-$path covering of the subgraph induced by $V(\mathcal{Q}_n) \setminus \mathcal{F}$ with one path joining a_i and b_i, for each $i \in \{1, \ldots, N+O\}$.*

In [CG], the smallest m defined in Problem 1.2 is denoted $[M, C, N, O]$. For example, using this notation, Havel's Lemma states $[0, 0, 1, 0] = 1$ and Lemma 1.1 states $[0, 0, 2, 0] = 2$. In the following table are given some of the values of $[M, C, N, O]$ known to us. Most of them were obtained in [CG]. The rows represent admissible combinations of M and C and the columns contain all the values of N and O such that $N + O \leq 3$. Each star in the table represents an impossible case. The missing entries in the table correspond to values of $[M, C, N, O]$ that we do not know yet. The inequalities in the table represent an upper or lower bound of the corresponding entry. The values with an asterisk represent results known to us that have not been published yet.

$MC \backslash NO$	01	10	20	11	02	30	21	12	03
00	$\star$	1	2	$\star$	4	5	$\star$	4^*	$\star$
11	2	$\star$	$\star$	4	$\star$	$\star$	5^*	$\star$	5^*
20	$\star$	4	4	$\star$	5		$\star$		$\star$
22	$\star$	$\star$	$\star$	$\star$	4	$\star$	$\star$	≤ 6	$\star$
31	4	$\star$	$\star$	5	$\star$	$\star$		$\star$	
33	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	≤ 6
40	$\star$	5	5^*	$\star$			$\star$		$\star$
42	$\star$	$\star$	$\star$	$\star$	5	$\star$	$\star$		$\star$
44	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
51	5	$\star$	$\star$	≥ 5	$\star$	$\star$		$\star$	

This paper continues the study of path coverings with prescribed ends in (faulty) hypercubes. More specifically, we prove that if $n \geq 4$ and $g_1, g_2, g_3 \in V(\mathcal{Q}_n)$ are distinct green vertices and $r_1, r_2, r_3 \in V(\mathcal{Q}_n)$ are distinct red vertices, then there exists a $3-$path covering of $\mathcal{Q}_n$ with paths joining g_1 and r_1, g_2 and g_3, and r_2 and r_3, respectively. In the proof we use many of the results mentioned in the above table. Also, we show that such a $3-$path covering does not always exist if $n = 3$. In the notation of [CG], this means that $[0, 0, 1, 2] = 4$ (which is the value in the first row and in the eight column of the table). In [CGGL] we provide a proof that $[4,0,2,0]=5$ and we use it to show that $[7,1,0,1]=6$. The proofs of the statements $[1,1,2,1]=5$ and $[1,1,0,3]=5$ will appear in a forthcoming paper.

2. Preliminaries

The three coordinates in $\mathcal{Q}_3$ will be denoted x, y and z, and any path with an endpoint r can be written uniquely as (r,η), where η is a sequence of letters in x, y, z (such a sequence will be called a *word*). For example, the path starting at $r=(0,1,0)$, going through $(1,1,0),(1,0,0)$ and ending at $(1,0,1)$ can be written (r, xyz). In $\mathcal{Q}_4$, the fourth coordinate will be denoted v.

The initial vertex of the path (r,η) is r; if the terminal vertex is s, then the same path will sometimes be written $(r,\eta; s)$.

For each vertex a in $\mathcal{Q}_n$ there is a unique vertex $\bar{a}$ in $\mathcal{Q}_n$ at distance n from a. The coordinates of $\bar{a}$ are the negation of the corresponding coordinates of a.

In many cases, it will be useful to view $\mathcal{Q}_n$ with $n \geq 4$ as two copies of Q_{n-1}, denoted $\mathcal{Q}_n^{top}$ and $\mathcal{Q}_n^{bot}$, with the vertices of $\mathcal{Q}_n^{top}$ and those of $\mathcal{Q}_n^{bot}$ joined by edges in pairs. $\mathcal{Q}_n^{top}$ and $\mathcal{Q}_n^{bot}$ will be called the *top plate* and the *bottom plate*, respectively. Without loss of generality, we may assume that the vertices of $\mathcal{Q}_n^{top}$ all share the same v-coordinate (say $v=1$), and the vertices of $\mathcal{Q}_n^{bot}$ all share the same v-coordinate (say $v=0$).

Given a subset $\mathcal{F}\subset V(\mathcal{Q}_n)$, we will denote $\mathcal{F}^{top}=V(\mathcal{Q}_n^{top})\cap\mathcal{F}$ and $\mathcal{F}^{bot}=V(\mathcal{Q}_n^{bot})\cap\mathcal{F}$.

We will allow path coverings to contain paths of length zero. A path of length zero is just a vertex a. In the list of paths of a path covering such a zero length path will be represented by (a) or by (a,ξ) where ξ is the empty word. The empty word will be denoted by $\emptyset$. The length of a word ξ is denoted by $|\xi|$. Also ξ^R means the reverse word of ξ; ξ' denotes the word obtained after the last letter is deleted from ξ; ξ^* is the word obtained after the first letter is deleted from ξ; $\varphi(\xi)$ is the first letter of ξ, and $\lambda(\xi)$ is the last letter of ξ.

If P_1, P_2 are path coverings of $\mathcal{Q}_n^{top}-\mathcal{F}^{top}$ and $\mathcal{Q}_n^{bot}-\mathcal{F}^{bot}$ respectively, and $a_1,\dots a_k$ are some terminals in P_1 such that $a_1v,\dots,a_kv$ are terminals in P_2, then $(P_1\star P_2; a_1,\dots,a_k)$ will denote the subgraph of $\mathcal{Q}_n$ obtained by adding the edges $\{a_1,a_1v\},\dots,\{a_k,a_kv\}$ to the union of the paths from P_1 and P_2. Note that $(P_1\star P_2; a_1,\dots,a_k)$ is a path covering of $\mathcal{Q}_n-\mathcal{F}$ if and only if it contains no cycles.

There is only one Hamiltonian cycle (up to isomorphism) in $\mathcal{Q}_3$: the cycle (a, μ), with $\mu = xyxzxyxz$. We refer to the word μ as the canonical representation of a Hamiltonian cycle in $\mathcal{Q}_3$.

Remark 2.1. *Let (a, ξ) be any Hamiltonian cycle in $\mathcal{Q}_3$. Then any two appearances of the same letter in ξ are separated by a word of odd length.*

3. On path coverings with conditions on the lengths of the paths

Proposition 3.1. *Let g_1 and g_2 be two green vertices and r be a red vertex in $\mathcal{Q}_3$. Then there exist at least two $2-$path coverings $P_1 : (g_1, \theta_1; g_2)$, (r, ξ_1) and $P_2 : (g_1, \theta_2; g_2)$, (r, ξ_2) of $\mathcal{Q}_3$ with $r\xi_1 \neq r\xi_2$.*

Proof. Without loss of generality we can assume that $g_2 = g_1xy$ and that $r = g_1x$ or $r = g_1z$. When $r = g_1x$ the words $\theta_1 = \theta_2 = yx$, $\xi_1 = zxyx$, $\xi_2 = zyxy$ satisfy the conditions, and when $r = g_1z$ the words $\theta_1 = xy$, $\theta_2 = yx$, $\xi_1 = xyxz$, $\xi_2 = yxyz$ satisfy the conditions. □

Definition 3.2. *Let $\{a, b\}$, $\{c, d\}$ be two pairs of vertices in $\mathcal{Q}_n$. We say that these pairs are parallel if there is a word ξ such that $b = a\xi$ and $d = c\xi$.*

Proposition 3.3. *Let r_1, r_2, g_1 and g_2 be two red and two green vertices in $\mathcal{Q}_3$. Then a $2-$path covering $(r_1, \xi; r_2)$, $(g_1, \eta; g_2)$ of $\mathcal{Q}_3$ exists if and only if the pairs $\{r_1, r_2\}$ and $\{g_1, g_2\}$ are not parallel.*

Proof. Assume that $\{r_1, r_2\}$ and $\{g_1, g_2\}$ are parallel and $(r_1, \xi; r_2)$, $(g_1, \eta; g_2)$ is a $2-$path covering of $\mathcal{Q}_3$. We can assume, without loss of generality, that $g_2 = g_1xy$ and $r_2 = r_1xy$. We can also assume that r_1 is adjacent to g_2. This means that there is a one-letter word θ such that $r_1 = g_2\theta$. This implies that $r_2 = r_1xy = g_2\theta xy = g_1xy\theta xy = g_1\theta$. Therefore, $(g_1, \eta\theta\xi\theta)$ should be a Hamiltonian cycle of $\mathcal{Q}_3$. But the word ξ is of even length and is separating two appearances of the letter θ. This contradicts Remark 2.1.

Assume now that $\{r_1, r_2\}$ and $\{g_1, g_2\}$ are not parallel. We can assume, without loss of generality, that $g_2 = g_1xy$ and $r_2 = r_1xz$. We can also assume that $r_1 = g_2x$. Then $(g_1, xy; g_2)$, $(r_1, zyxy; r_2)$ is the desired $2-$path covering of $\mathcal{Q}_3$. □

Proposition 3.4. *Let g, g_1 and g_2 be three green vertices and r be a red vertex in $\mathcal{Q}_3$. Then there exist at least two $3-$path coverings $P_i : (g_1, \theta_i; g_2)$, (r, ξ_i), (g, η_i) of $\mathcal{Q}_3$ such that for each $i = 1, 2$, $|\xi_i|$ is even, $|\eta_i|$ is odd, and $g\eta_1 \neq g\eta_2$.*

Proof. Without loss of generality, we may assume that $g_2 = g_1xy$ and $g = g_1xz$. Up to isomorphism, there are three different possibilities for $r : gxyz, gz, gx$.

The following tables provide examples of path coverings that satisfy the conditions of the proposition for each of the values of r.

r	θ	ξ	η
$gxyz$	xy	zx	x
	xy	zy	y
	xy	$\emptyset$	yxy
	xy	$\emptyset$	xyx
	$zyxz$	$\emptyset$	z

r	θ	ξ	η
gz	yx	$\emptyset$	xyx
	yx	$\emptyset$	yxy
	$yzxz$	$\emptyset$	x
	$zyzx$	$\emptyset$	y

r	θ	ξ	η
gx	xy	yz	y
	xy	$\emptyset$	yxz
	yx	yx	z
	$yzxz$	$\emptyset$	z

□

Proposition 3.5. *Let r_1, r_2, g_1, g_2 and g_3 be two distinct red vertices and three distinct green vertices in $\mathcal{Q}_3$. Then there exists a path covering $P : (r_1, \xi; g_1)$, (r_2, μ), (g_2), (g_3, η), where μ is of even length and η is of odd length.*

Proof. Without loss of generality we assume that $g_1 = r_1xyz$ or that $g_1 = r_1x$.

If $g_1 = r_1xyz$, we can assume, again without loss of generality, that $g_2 = r_1x$ and $g_3 = r_1y$. The following table gives the paths for the three possible values of r_2.

r_2	ξ	μ	η
r_1xy	zxy	$\emptyset$	z
r_1xz	zyx	$\emptyset$	x
r_1yz	zxy	$\emptyset$	x

If $g_1 = r_1x$, we can assume, again without loss of generality, that $g_2 = r_1y$ or $g_2 = r_1xyz$. The following table contains the paths for all the possible values of r_2 and g_3.

g_2	r_2	g_3	ξ	μ	η
r_1y	r_1xy	r_1z	x	$\emptyset$	xyx
r_1y	r_1xy	r_1xyz	x	$\emptyset$	xyx
r_1y	r_1xz	r_1z	x	$\emptyset$	yxz
r_1y	r_1xz	r_1xyz	x	xy	z
r_1y	r_1yz	r_1z	x	$\emptyset$	xyz
r_1y	r_1yz	r_1xyz	x	yx	z
r_1xyz	r_1xy	r_1y	x	$\emptyset$	zyx
r_1xyz	r_1xy	r_1z	x	xz	x
r_1xyz	r_1xz	r_1y	x	xy	x
r_1xyz	r_1xz	r_1z	x	$\emptyset$	yzx
r_1xyz	r_1yz	r_1y	x	yx	x
r_1xyz	r_1yz	r_1z	x	zx	x

□

4. THE COTERMINAL SET OF A HYPERCUBE-VALUED VECTOR

Let $X = (a_1, a_2, \ldots, a_k)$ be a $k-$tuple of distinct vertices in $\mathcal{Q}_n$. We say that $Y = (b_1, \ldots, b_k)$ is coterminal to X and we shall denote that by $X\sharp Y$, if there exists a $k-$path covering $P : (a_1, \xi_1; b_1), \ldots, (a_k, \xi_k; b_k)$. The set of all $k-$tuples that are coterminal to X is denoted by X^{cot}. For example, if $X = (a)$ then X^{cot} is the set of all vertices whose parity is opposite to a's. If $n \geq 2$ and $X = (r, g)$, where r and g are respectively a red vertex and a green vertex, then X^{cot} contains all ordered pairs (g_1, r_1) with $g_1 \neq g$ and $r_1 \neq r$ (since $[0, 0, 2, 0] = 2$, as we know from Lemma 1.1). X^{cot} also contains all ordered pairs of the form (r, g_1) with $g_1 \neq g$ and all ordered pairs (r_1, g) with $r_1 \neq r$ (since $[1, 1, 0, 1] = 2$, as proved in [CG]). If $n = 3$ then it follows from Proposition 3.3 that $(r, g)^{cot}$ also contains all ordered pairs (r_1, g_1) that are not parallel to (r, g).

Proposition 4.1. *Let (r, g) be any pair of a red vertex and a green vertex in $\mathcal{Q}_3$, and let (g', xy) be any path of length two in $\mathcal{Q}_3$ that starts with a green vertex g' and does not contain g. Then*

$$\{(g', g'x), (g'x, g'xy)\} \cap (r, g)^{cot} \neq \emptyset.$$

Proof. If $(g', g'x) \notin (r, g)^{cot}$ then $g'x = r$ since $g' \neq g$. But then $(g'x, g'xy) = (r, ry) \in (r, g)^{cot}$ for $ry \neq g$. □

Proposition 4.2. *Let (g, r) be any pair of a green vertex and a red vertex in $\mathcal{Q}_3$, and let (g', ξ) be any path of length three in $\mathcal{Q}_3$ that starts with a green vertex g' and does not contain g and r as consecutive vertices. Then there exist words μ, ν with $\xi = \mu\nu$ such that $(g'\mu, g'\mu\varphi(\nu)) \in (g, r)^{cot}$.*

Proof. The proof is by contradiction. Let $\xi = x_1x_2x_3$ and assume that none of the pairs $(g', g'x_1)$, $(g'x_1, g'x_1x_2)$, $(g'x_1x_2, g'x_1x_2x_3)$ is in $(g, r)^{cot}$. The assumption that $(g', g'x_1) \notin (g, r)^{cot}$ and the hypothesis that g and r are not consecutive vertices in the path (g', ξ) imply that $g' \neq g, g'x_1 \neq r$ and that $(g', g'x_1)$ is parallel to (g, r). Hence, $r = gx_1$. Now the assumption $(g'x_1, g'x_1x_2) \notin (g, r)^{cot}$ and the fact $g'x_1 \neq r$ imply that $g'x_1x_2 = g$. Finally, the assumption $(g'x_1x_2, g'x_1x_2x_3) = (g, gx_3) \notin (g, r)^{cot}$ implies that $gx_3 = r$, which contradicts the hypothesis that g and r are not consecutive vertices in the path (g', ξ). □

Proposition 4.3. *Let (r, g) and $(\hat{r}, \hat{g})$ be two not necessarily distinct pairs of green and red vertices in $\mathcal{Q}_3$. Let also ξ, η, and θ be any three nonempty distinct even words. Then*

$$\{(r, g\xi), (r\eta, g), (r\xi, g\theta)\} \cap (\hat{g}, \hat{r})^{cot} \neq \emptyset.$$

Proof. If neither $(r, g\xi)$ nor $(r\eta, g)$ is in $(\hat{g}, \hat{r})^{cot}$ then either $r = \hat{r}$ and $g = \hat{g}$ or else $r\eta = \hat{r}$ and $g\xi = \hat{g}$. Then $r\xi \neq \hat{r}$ and $g\theta \neq \hat{g}$. Consequently, $(r\xi, g\theta) \in (\hat{g}, \hat{r})^{cot}$. □

Proposition 4.4. *Let (r, g) and $(\hat{r}, \hat{g})$ be two (not necessarily distinct) pairs of vertices in $\mathcal{Q}_3$. Let also ξ be any nonempty even word, and μ, ν be any two odd words such that $r\mu \neq r\nu$. Then*

$$\{(r, g\xi), (r\xi, g), (r\mu, g\nu)\} \cap (\hat{g}, \hat{r})^{cot} \neq \emptyset.$$

Proof. If neither $(r, g\xi)$ nor $(r\xi, g)$ is in $(\hat{g}, \hat{r})^{cot}$ then either $r = \hat{r}$ and $g = \hat{g}$ or else $r\xi = \hat{r}$ and $g\xi = \hat{g}$. In either case (r, g) is parallel to $(\hat{r}, \hat{g})$. It follows then that $(r\mu, g\nu)$ is not parallel to $(\hat{g}, \hat{r})$ and therefore, by Proposition 3.3, $(r\mu, g\nu) \in (\hat{g}, \hat{r})^{cot}$. □

Proposition 4.5. *Let g_1, g_2 and r_1, r_2, be two distinct green and two distinct red vertices in $\mathcal{Q}_3$. Let also $(\hat{g}, \hat{r})$ be any pair of a green vertex and a red vertex in $\mathcal{Q}_3$. Then there exists a $3-$path covering $P : (r_1, \xi; g_1), (r_2, \zeta), (g_2, \theta)$ such that $(r_2\zeta, g_2\theta) \in (\hat{g}, \hat{r})^{cot}$.*

Proof. Without loss of generality, we assume that $r_2 = r_1xy$. Then, up to isomorphism, there are two possibilities for g_1 : (1) $g_1 = r_1x$; or (2) $g_1 = r_1xyz$. The following two tables give examples of $3-$path coverings $(r_1, \xi; g_1), (r_2, \zeta), (g_2, \theta)$ for the cases (1) and (2) and all the possible nonequivalent values of g_2 for each case. These examples are sufficient to show that for each case we can choose three different values of $(r_2\zeta, g_2\theta)$ that are suitable for the application of Proposition 4.3 or Proposition 4.4 as explained below.

g_1	g_2	ξ	ζ	θ
r_1x	r_1y	x	$\emptyset$	$zxyx$
			$zxyx$	$\emptyset$
			z	zyx
	r_1z	x	$\emptyset$	$xyxz$
			$xzxy$	$\emptyset$
			x	xyx
	r_1xyz	x	$\emptyset$	$yxyz$
			$xzyx$	$\emptyset$
			xz	yx

g_1	g_2	ξ	ζ	θ
r_1xyz	r_1x	yzx	$\emptyset$	zx
		zxy	xz	$\emptyset$
		zyx	x	z
	r_1z	xzy	$\emptyset$	yz
			xz	$\emptyset$
			x	y
		yzx	$\emptyset$	xz
			yz	$\emptyset$
			y	x

Now, for the case $g_1 = r_1x, g_2 = r_1y$ the three values of $(r_2\zeta, g_2\theta)$ produced in the table form the set $\{(r_2, g_2zxyx), (r_2zxyx, g_2), (r_2z, g_2zyx)\}$, which simplifies to $\{(r_2, g_2zy), (r_2zy, g_2), (r_2z, g_2zyx)\}$, and therefore, Proposition 4.4 applies with $r = r_2, g = g_2, \xi = zy, \mu = z, \nu = zyx$. In the following table we list all the cases indicating the set formed by the selected values of $(r_2\zeta, g_2\theta)$ and the proposition to be used with $r = r_2, g = g_2$ and the words indicated there.

g_1	g_2	$(r_2\zeta, g_2\theta)$-set	Proposition to be applied
r_1x	r_1y	$\{(r_2, g_2zy), (r_2zy, g_2), (r_2z, g_2zyx)\}$	4.4 with $\xi = zy$, $\mu = z, \nu = zyx$.
r_1x	r_1z	$\{(r_2, g_2yz), (r_2yz, g_2), (r_2x, g_2y)\}$	4.4 with $\xi = yz$, $\mu = x, \nu = y$.
r_1x	r_1xyz	$\{(r_2, g_2xz), (r_2yz, g_2), (r_2xz, g_2yx)\}$	4.3 with $\xi = xz$, $\mu = yz, \nu = yx$.
r_1xyz	r_1x	$\{(r_2, g_2xz), (r_2xz, g_2), (r_2x, g_2z)\}$	4.4 with $\xi = xz$, $\mu = x, \nu = z$.
r_1xyz	r_1z	$\{(r_2, g_2yz), (r_2yz, g_2), (r_2x, g_2y)\}$	4.4 with $\xi = yz$, $\mu = x, \nu = y$.

□

5. PATH COVERINGS WITH A LONG PATH

The following lemma complements Lemma 3.14 in [CG].

Lemma 5.1. *Let $n \geq 3$, and r_1, r_2 and g_1, g_2 be two red and two green vertices in $\mathcal{Q}_n$. If $n = 3$ we also assume that r_2 and g_2 are adjacent in $\mathcal{Q}_3$ and $(r_1, g_1) \neq (\overline{g_2}, \overline{r_2})$. Then there exists a 2−path covering $(r_1, \xi; g_1)$, $(r_2, \eta; g_2)$ of $\mathcal{Q}_n$ such that $|\xi| > 2^{n-1}$.*

Proof. Let $n = 3$. Without loss of generality we can assume that $r_2 = r_1xy$ and that either $g_2 = r_1y$ or $g_2 = r_1xyz$. The following table shows the desired path coverings for all nonequivalent admissible positions of g_1.

g_2	g_1	ξ	η
r_1y	r_1x	$zyxyz$	x
	r_1z	$xzyxy$	x
	r_1xyz	$xzxyx$	x
r_1xyz	r_1x	$yzyxz$	z

Let $n \geq 4$. We continue the proof by induction on n. First, we split $\mathcal{Q}_n$ into two plates so that $r_1 \in \mathcal{Q}_n^{top}$ and $r_2 \in \mathcal{Q}_n^{bot}$. There are four cases to consider that depend on the distribution of g_1, g_2 among the plates.

Case 1. g_1 and g_2 are on the top plate.

Choose a red vertex $r_3 \neq r_1$ in $\mathcal{Q}_n^{top}$. If $n = 4$ we additionally require that (a) r_3 be a neighbor of g_2; (b) r_3v be a neighbor of r_2; and (c) $(r_1, g_1) \neq (\overline{g_2}, \overline{r_3})$ (here the complements are taken in $\mathcal{Q}_n^{top}$). By the induction hypothesis, there is a 2−path covering $(r_1, \xi; g_1)$, $(g_2, \eta; r_3)$ of $\mathcal{Q}_n^{top}$ with $|\xi| > 2^{n-2}$. Let μ and ν be such words that $|\mu|$ is odd, $\xi = \mu\nu$, $r_1\mu v \neq r_2$, and if $n = 4$ also $r_1\mu v \neq \overline{r_3 v}$, hence $(r_1\mu' v, r_1\mu v) \neq (\overline{r_2}, \overline{r_3 v})$ (here the complements are taken in $\mathcal{Q}_n^{bot}$). Such words μ and ν exists since $n \geq 4$ and $|\xi| \geq 2^{4-2} + 1 = 5$ and therefore there are at least three odd vertices in $(r_1, \xi; g_1)$. Again by the induction hypothesis, there exists a $2-$path covering $(r_1\mu' v, \zeta; r_1\mu v)$, $(r_3 v, \theta; r_2)$ of $\mathcal{Q}_n^{bot}$ with $|\zeta| > 2^{n-2}$. The desired $2-$path covering of $\mathcal{Q}_n$ is $(r_1, \mu' v\zeta v\nu)$, $(g_2, \eta v\theta; r_2)$.

Case 2. g_1 is on the top plate and g_2 is on the bottom plate.

Let $(r_1, \xi; g_1)$ be a Hamiltonian path of $\mathcal{Q}_n^{top}$. There exist words μ and ν, with $|\mu| \geq 1$, such that neither $r_1\mu' v$ nor $r_1\mu v$ is in $\{r_2, g_2\}$. Let $(r_1\mu' v, \zeta; r_1\mu v)$, $(r_2, \eta; g_2)$ be a $2-$path covering of $\mathcal{Q}_n^{bot}$. The desired $2-$path covering of $\mathcal{Q}_n$ is $(r_1, \mu' v\zeta v\nu; g_1)$, $(r_2, \eta; g_2)$.

Case 3. g_2 is on the top plate and g_1 is on the bottom plate.

If $g_2 v = r_2$ choose any red vertex $r_3 \neq r_1$ in the top plate such that $r_3 v \neq g_1$. Use $[1, 1, 0, 1] = 2$ to find a Hamiltonian path $(r_1, \xi; r_3)$ of $\mathcal{Q}_n^{top} - \{g_2\}$ and a Hamiltonian path $(r_3 v, \eta; g_1)$ of $\mathcal{Q}_n^{bot} - \{r_2\}$. The desired $2-$path covering of $\mathcal{Q}_n$ is $(r_1, \xi v\eta; g_1)$, $(g_2, v; r_2)$. If $g_2 v \neq r_2$ choose any red vertex $r_3 \neq r_1$ in the top plate such that (a) $r_3 v \neq g_1$; and (b) the pairs $(g_1, r_3 v)$ and $(g_2 v, r_2)$ are not parallel in $\mathcal{Q}_n^{bot}$. Use $[1, 1, 0, 1] = 2$ to find a Hamiltonian path $(r_1, \xi; r_3)$ of $\mathcal{Q}_n^{top} - \{g_2\}$ and use $[0, 0, 0, 2] = 4$ or, if $n = 4$, Proposition 3.3 to find a $2-$path covering $(r_3 v, \eta; g_1)$, $(g_2 v, \zeta; r_2)$ of $\mathcal{Q}_n^{bot}$. The desired $2-$path covering of $\mathcal{Q}_n$ is $(r_1, \xi v\eta; g_1)$, $(g_2, v\zeta; r_2)$.

Case 4. g_1 and g_2 are on the bottom plate.

Let g_3 be a green vertex in $\mathcal{Q}_n^{top}$ such that $g_3 v \neq r_2$ and $(r_1, \xi; g_3)$ be a Hamiltonian path of $\mathcal{Q}_n^{top}$. There exists a $2-$path covering $(r_2, \eta; g_2)$, $(g_3 v, \zeta; g_1)$ of $\mathcal{Q}_n^{bot}$ for $[0, 0, 2, 0] = 2$. The desired $2-$path covering of $\mathcal{Q}_n$ is $(r_1, \xi v\zeta; g_1)$, $(r_2, \eta; g_2)$. □

6. $[0, 0, 1, 2] = 4$

In this section we prove the main result in this paper.

Theorem 6.1 ($[0, 0, 1, 2] = 4$)**.** *Let $n \geq 4$ and $r_1, r_2, r_3, g_1, g_2, g_3$ be three red and three green vertices in $\mathcal{Q}_n$. Then there exists a $3-$path covering $(r_1, \xi; g_1)$, $(r_2, \eta; r_3)$, $(g_2, \zeta; g_3)$ of $\mathcal{Q}_n$. The claim is not true if $n = 3$.*

Proof. Let $n = 3$ and r_1 be any red vertex in $\mathcal{Q}_3$. Let also $r_2 = r_1 xy$, $r_3 = r_1 yz$, $g_1 = r_1 xyz$, $g_2 = r_1 y$ and $g_3 = r_1 x$. Then a $3-$path covering $(r_1, \xi; g_1)$, $(r_2, \eta; r_3)$, $(g_2, \zeta; g_3)$ of $\mathcal{Q}_3$ does not exist.

Now, let $n \geq 4$. In the first part of the proof we consider the case $n = 4$. In the second part we use mathematical induction to prove our claim for every $n > 4$.

Part 1. Let $n = 4$. Then r_1 and g_1 coincide exactly at one or at three coordinates.

A. Every coordinate that does not separate r_1 from g_1 also does not separate r_2 from r_3 and g_2 from g_3.

Since r_2 (g_2) differ from r_3 (g_3) at least at two coordinates, this case is only possible when r_1 and g_1 differ at three coordinates. We split $\mathcal{Q}_4$ into two plates using the only coordinate where r_1 and g_1 coincide. Without loss of generality we can assume that r_1 and g_1 are on the top plate and $r_1 = g_1xyz$.

Case 1. r_2, r_3, g_2, g_3 are on the top plate.

Without loss of generality we can assume that $r_2 = g_1x$ and $r_3 = g_1y$. There are two possible cases to consider.

(i) $g_2 = g_1xz$, $g_3 = g_1yz$.

Let $(g_1v, \xi; r_1v)$ be a Hamiltonian path of $\mathcal{Q}_4^{bot}$. Then the desired $3-$path covering of $\mathcal{Q}_4$ is $(g_1, v\xi v; r_1)$, $(r_2, yx; r_3)$, $(g_2, xy; g_3)$.

(ii) $g_2 = g_1xz$, $g_3 = g_1xy$.

Clearly (r_2, r_3) and (g_2, g_3) are not parallel in $\mathcal{Q}_4^{top}$. Therefore (r_2v, r_3v) and (g_2v, g_3v) are not parallel in $\mathcal{Q}_4^{bot}$. Then, according to Proposition 3.3, there exists a $2-$path covering $(r_2v, \xi; r_3v)$, $(g_2v, \eta; g_3v)$ of $\mathcal{Q}_4^{bot}$. Then the desired $3-$path covering of $\mathcal{Q}_4$ is $(g_1, zyx; r_1)$, $(r_2, v\xi v; r_3)$, $(g_2, v\eta v; g_3)$.

Case 2. r_2, r_3 are on the top plate, g_2, g_3 are on the bottom plate.

Without loss of generality we can assume that $r_2 = g_1x$ and $r_3 = g_1y$. There are three possible cases to consider.

(i) $g_2 = g_1vz$, $g_3 = r_1v$.

Then a $3-$path covering of $\mathcal{Q}_4$ is $(g_1, vyxyzvxyx; r_1)$, $(r_2, yx; r_3)$, $(g_2, yx; g_3)$.

(ii) $g_2 = g_1vy$, $g_3 = g_1vx$.

Then a $3-$path covering of $\mathcal{Q}_4$ is $(g_1, vzyxyvxyx; r_1)$, $(r_2, yx; r_3)$, $(g_2, xy; g_3)$.

(iii) $g_2 = g_1vx$, $g_3 = r_1v$.

Then a $3-$path covering of $\mathcal{Q}_4$ is $(g_1, vyzyxvxyx; r_1)$, $(r_2, yx; r_3)$, $(g_2, yz; g_3)$.

Case 3. r_2, r_3, g_2, g_3 are on the bottom plate.

There are two possible cases to consider.

(i) (r_2, r_3) and (g_2, g_3) are not parallel in $\mathcal{Q}_4^{bot}$.

According to Proposition 3.3 there exists a $2-$path covering $(r_2, \xi; r_3)$, $(g_2, \eta; g_3)$ of $\mathcal{Q}_4^{bot}$. Let also $(g_1, \zeta; r_1)$ be a Hamiltonian path of $\mathcal{Q}_4^{top}$. The desired $3-$path covering of $\mathcal{Q}_4$ is $(g_1, \zeta; r_1)$, $(r_2, \xi; r_3)$, $(g_2, \eta; g_3)$.

(ii) (r_2, r_3) and (g_2, g_3) are parallel in $\mathcal{Q}_4^{bot}$.

There are three possible cases to consider.

(a) $r_2 = g_1v$, $r_3 = r_1vy$, $g_2 = g_1vz$, $g_3 = g_1vx$.

Then a 3−path covering of $\mathcal{Q}_4$ is $(g_1, yzx; r_1)$, $(r_2, yzxy; r_3)$, $(g_2, vxzyvy; g_3)$.

(b) $r_2 = g_1v$, $r_3 = r_1vy$, $g_2 = g_1vy$, $g_3 = r_1v$.

Then a 3−path covering of $\mathcal{Q}_4$ is $(g_1, xvyvxzyxy; r_1)$, $(r_2, zx; r_3)$, $(g_2, zx; g_3)$.

(c) $r_2 = r_1vz$, $r_3 = r_1vy$, $g_2 = g_1vy$, $g_3 = g_1vz$.

Then a 3−path covering of $\mathcal{Q}_4$ is $(g_1, vxvyxzyxy; r_1)$, $(r_2, zy; r_3)$, $(g_2, zy; g_3)$.

B. There exists a coordinate that does not separate r_1 from g_1 and separates r_2 from r_3 or g_2 from g_3.

We split $\mathcal{Q}_4$ into two plates using that coordinate. Without loss of generality we can assume that r_2 and r_3 are separated and that r_2 is in $\mathcal{Q}_4^{top}$. There are three possible cases to consider.

Case 1. g_2, g_3 are on the top plate.

We can assume that $g_2v \neq r_3$ by renumbering g_2 and g_3, if necessary. Using Proposition 3.5 one can find a 4−path covering $P_1 : (r_1, \xi; g_1)$, (r_2, μ), (g_2), (g_3, η) of $\mathcal{Q}_4^{top}$, where μ is a word of even length and η is a word of odd length. Also, there exists a 2−path covering $P_2 : (r_2\mu v, \nu; r_3)$, $(g_2v, \zeta; g_3\eta v)$ of $\mathcal{Q}_4^{bot}$ for $[0, 0, 2, 0] = 2$. The desired 3−path covering of $\mathcal{Q}_4$ is $(P_1 \star P_2; r_2\mu, g_2, g_3\eta)$.

Case 2. g_2, g_3 are on the bottom plate.

There exists exactly one red vertex r_4 in $\mathcal{Q}_4^{bot}$ such that (r_3, r_4) is parallel to (g_2, g_3). There are exactly four green vertices in $\mathcal{Q}_4^{top}$. Therefore there exists at least on green vertex g_4 in $\mathcal{Q}_4^{top}$ which is different from g_1, r_3v and r_4v. There exists a 2−path covering $(r_1, \xi; g_1)$, $(r_2, \eta; g_4)$ of $\mathcal{Q}_4^{top}$ for $[0, 0, 2, 0] = 2$. Since (r_3, g_4v) and (g_2, g_3) are not parallel, it follows from Proposition 3.3 that there exists a 2−path covering $(g_4v, \nu; r_3)$, $(g_2, \mu; g_3)$ of $\mathcal{Q}_4^{bot}$. The desired 3−path covering of $\mathcal{Q}_4$ is $(r_1, \xi; g_1)$, $(r_2, \eta v\nu; r_3)$, $(g_2, \mu; g_3)$.

Case 3. g_2 is on the top plate, g_3 is on the bottom plate.

In this case Proposition 4.5 guarantees that there exists a path covering $P_1 : (r_1, \xi; g_1)$, (r_2, ζ), (g_2, θ) of $\mathcal{Q}_4^{top}$ such that, relative to $\mathcal{Q}_4^{bot}$, $(r_2\zeta v, g_2\theta v) \in (r_3, g_3)^{cot}$. Let $P_2 : (r_2\zeta v, \mu; r_3)$, $(g_2\theta v\nu; g_3)$ be a 2−path covering of $\mathcal{Q}_4^{bot}$. The desired 3−path covering of $\mathcal{Q}_4$ is $(P_1 \star P_2; r_2\zeta, g_2\theta)$.

Part 2. Now we shall prove our claim for every $n > 4$. The proof is by induction. We assume that $n > 4$ and that our claim is true for every integer greater than or equal to four and less than or equal to $n - 1$. We shall prove our claim for n.

We split $\mathcal{Q}_n$ into two plates such that $r_2 \in \mathcal{Q}_n^{top}$ and $r_3 \in \mathcal{Q}_n^{bot}$. Without loss of generality we can also assume that g_2 is on the top plate. We have to consider eigth cases that correspond to the different distributions of the vertices r_1, g_1 and g_3 among the plates.

Case 1. r_1, g_1, g_3 are on the top plate.

Let r^* be any red vertex in $\mathcal{Q}_n^{top}$ different from r_1 and r_2. By the induction hypothesis there exists a path covering $P_1 : (r_1, \xi; g_1), (g_2, \eta; g_3), (r_2, \theta; r^*)$ of $\mathcal{Q}_n^{top}$. Let $P_2 : (r^*v, \zeta; r_3)$ be a Hamiltonian path of $\mathcal{Q}_n^{bot}$. The desired $3-$path covering of $\mathcal{Q}_n$ is $(P_1 \star P_2; r^*)$.

Case 2. r_1, g_1 are on the top plate and g_3 is on the bottom plate.

Let r^* be any red vertex in $\mathcal{Q}_n^{top}$ different from r_1 and r_2 and such that $r^*v \neq g_3$. Let also g^* be a green vertex in $\mathcal{Q}_n^{top}$ different from g_1 and g_2 and such that $g^*v \neq r_3$. By the induction hypothesis there exists a path covering $P_1 : (r_1, \xi; g_1)$, $(g_2, \eta; g^*)$, $(r_2, \theta; r^*)$ of $\mathcal{Q}_n^{top}$. Since $[0, 0, 2, 0] = 2$, there exists a $2-$path covering $P_2 : (g^*v, \mu; g_3)$, $(r^*v, \nu; r_3)$ of $\mathcal{Q}_n^{bot}$. The desired $3-$path covering of $\mathcal{Q}_n$ is $(P_1 \star P_2; g^*, r^*)$.

Case 3. r_1, g_3 are on the top plate and g_1 is on the bottom plate.

Let r^* be any red vertex in $\mathcal{Q}_n^{top}$ different from r_1 and r_2 and such that $r^*v \neq g_1$. Let also g^* be a green vertex in $\mathcal{Q}_n^{top}$ different from g_2 and g_3 and such that $g^*v \neq r_3$. By the induction hypothesis there exists a path covering $P_1 : (r_1, \xi; g^*)$, $(g_2, \eta; g_3)$, $(r_2, \theta; r^*)$ of $\mathcal{Q}_n^{top}$. Since $[0, 0, 2, 0] = 2$, there exists a $2-$path covering $P_2 : (g^*v, \mu; g_1)$, $(r^*v, \nu; r_3)$ of $\mathcal{Q}_n^{bot}$. The desired $3-$path covering of $\mathcal{Q}_n$ is $(P_1 \star P_2; g^*, r^*)$.

Case 4. g_1, g_3 are on the top plate and r_1 is on the bottom plate.

Let r', r^* be two distinct red vertices in $\mathcal{Q}_n^{top}$ different from r_2. By the induction hypothesis there exists a path covering $P_1 : (g_1, \xi; r'), (g_2, \eta; g_3), (r_2, \theta; r^*)$ of $\mathcal{Q}_n^{top}$. Since $[0, 0, 2, 0] = 2$, there exists a $2-$path covering $P_2 : (r'v, \mu; r_1)$, $(r^*v, \nu; r_3)$ of $\mathcal{Q}_n^{bot}$. The desired $3-$path covering of $\mathcal{Q}_n$ is $(P_1 \star P_2; r', r^*)$.

Case 5. r_1 is on the top plate and g_1, g_3 are on the bottom plate.

Let g^* be a green vertex in $\mathcal{Q}_n^{top}$ different from g_2 such that $g^*v \neq r_3$. By Lemma 5.1, there exists a $2-$path covering $P_1 : (r_1, \xi; g_2), (r_2, \eta; g^*)$ of $\mathcal{Q}_n^{top}$ such that $|\xi| > 2^{n-2}$. Since $n \geq 5$, there exist words μ and ν, with $|\mu|$ odd, such that $\xi = \mu\nu$, $r_1\mu'v \neq g_1$, $r_1\mu'v \neq g_3$ and $g_2\nu^Rv \neq r_3$. Then, by the induction hypothesis, there exists a path covering $P_2 : (r_1\mu'v, \theta; g_1)$, $(g_2\nu^Rv, \rho; g_3)$, $(g^*v, \gamma; r_3)$ of $\mathcal{Q}_n^{bot}$. The desired $3-$path covering of $\mathcal{Q}_n$ is $(P_1 \star P_2; r_1\mu', g_2\nu^R, g^*)$.

Case 6. g_1 is on the top plate and r_1, g_3 are on the bottom plate.

This case is equivalent to **Case 5**.

Case 7. g_3 is on the top plate and r_1, g_1 are on the bottom plate.

Let r^* be a red vertex in $\mathcal{Q}_n^{top}$ different from r_2 and such that $r^*v \neq g_1$. Since $[0,0,2,0] = 2$, there exists a $2-$path covering $P_1 : (g_2, \eta; g_3), (r_2, \theta; r^*)$ of $\mathcal{Q}_n^{top}$ and a $2-$path covering $P_2 : (r_1, \xi; g_1)$, $(r^*v, \zeta; r_3)$ of $\mathcal{Q}_n^{bot}$. The desired $3-$path covering of $\mathcal{Q}_n$ is $(P_1 \star P_2; r^*)$.

Case 8. r_1, g_1, g_3 are on the bottom plate.

This case is equivalent to **Case 2**. □

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