THE NON-HAUSDORFF NUMBER OF A TOPOLOGICAL SPACE

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Abstract. We call a non-empty subset A of a topological space X *finitely non-Hausdorff* if for every non-empty finite subset F of A and every family $\{U_x : x \in F\}$ of open neighborhoods U_x of $x \in F$, $\cap\{U_x : x \in F\} \neq \emptyset$ and we define *the non-Hausdorff number* $nh(X)$ *of* X as follows: $nh(X) := 1 + \sup\{|A| : A$ is a finitely non-Hausdorff subset of $X\}$.

Using this new cardinal function we show that the following three inequalities are true for every topological space X

(i) $|X| \leq 2^{2^{d(X)}} \cdot nh(X)$;
(ii) $w(X) \leq 2^{(2^{2^{d(X)}} \cdot nh(X))}$;
(iii) $|X| \leq d(X)^{\chi(X)} \cdot nh(X)$

and

(iv) $|X| \leq nh(X)^{\chi(X)L(X)}$

is true for every T_1-topological space X, where $d(X)$ is the density, $w(X)$ is the weight, $\chi(X)$ is the character and $L(X)$ is the Lindelöf degree of the space X.

The first three inequalities extend to the class of all topological spaces Pospišil's inequalities that for every Hausdorff space X, $|X| \leq 2^{2^{d(X)}}$, $w(X) \leq 2^{2^{2^{d(X)}}}$ and $|X| \leq d(X)^{\chi(X)}$. The fourth inequality generalizes to the class of all T_1-spaces Arhangel′skiĭ's inequality that for every Hausdorff space X, $|X| \leq 2^{\chi(X)L(X)}$. It is still an open question if Arhangel′skiĭ's inequality is true for all T_1-spaces. It follows from (iv) that the answer of this question is in the afirmative for all T_1-spaces with $nh(X)$ not greater than the cardianilty of the continuum.

Examples are given to show that the upper bounds in (i) and (iii) are exact and that $nh(X)$ cannot be omitted.

1. Introduction

Let X be a topological space and for $x \in X$ let $\mathcal{N}_x$ denote the family of all open neighborhoods of x in X. For a nonempty subset A of X we denote by $\mathcal{U}_A$ the set of all families $\mathcal{U} := \{U_a : a \in A, U_a \in \mathcal{N}_a\}$. A family Γ is *centered* if the intersection of any finitely many elements of Γ is non-empty.

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This paper is respectfully dedicated to W. W. Comfort on the occasion of his 80th birthday.

Recall that $d(X) := \min\{|A| : A \subset X, \overline{A} = X\}$, $w(X) := \min\{|\mathcal{B}| : \mathcal{B}$ is a base for $X\}$, $L(X) := \min\{\kappa :$ every open cover of X has a subcover of cardinality $\leq \kappa\}$, $\chi(x, X) := \min\{|\mathcal{V}| : \mathcal{V}$ is a local base for $x\}$ and $\chi(X) := \sup\{\chi(x, X) : x \in X\}$.

In 1937, Pospišil proved (see [9]) that for every Hausdorff space X,

(a) $|X| \leq 2^{2^{d(X)}}$;
(b) $w(X) \leq 2^{2^{2^{d(X)}}}$; and
(c) $|X| \leq d(X)^{\chi(X)}$;

and in 1969, Arhangel′skiĭ (see [1]) answered a question of Alexandroff and Urysohn raised in 1923 by showing that for every Hausdorff space X, $|X| \leq 2^{\chi(X)L(X)}$. Since then many mathematicians have obtained similar inequalities for different classes of topological spaces but it is still unknown if Arhangel′skiĭ's inequality is true for all T_1-topological spaces (see the survey paper [8]).

In this paper we generalize Pospišil's inequalities for the class of all topological spaces and Arhangel′skiĭ's inequality for the class of all T_1-topological spaces and show that Arhangel′skiĭ's inequality is true for a very large class of T_1-spaces.

2. The cardinal function $nh(X)$

We begin with an example showing that Pospišil's inequality (c) is not always true for T_1-spaces.

Example 2.1. *Let $\mathbb{N}$ denote the set of all positive integers and $\mathbb{R}$ be the set of all real numbers. Let $S := \{1/n : n \in \mathbb{N}\}$ and $M := S \cup \{0\}$ be the subspace of $\mathbb{R}$ with the inherited topology. Then in M all points except 0 are isolated and $\lim_{n\to\infty} 1/n = 0$. Let α be an initial ordinal. We duplicate α many times the point $0 \in M$ i.e. we replace in M the point 0 with α many distinct points and obtain the set $X := S \cup \alpha$ with topology such that for each $\beta < \alpha$ we have $\lim_{n\to\infty} 1/n = \beta$ and the subspaces S and α with the inherited topology from X are discrete. Then the set $\{1/n : n \in \mathbb{N}\}$ is dense in X (hence $d(X) = \omega$), $\chi(X) = \omega$, and if $\alpha > 2^\omega$ then $|X| > d(X)^{\chi(X)} = \omega^\omega = 2^\omega$.*

To be able to generalize Pospišil's and Arhangel′skiĭ's inequalities we need to introduce some new concepts.

Definition 2.2. *We will call a nonempty subset A of a topological space X* finitely non-Hausdorff *if for every non-empty finite subset F of A and every $\mathcal{U} \in \mathcal{U}_F$, $\cap\mathcal{U} \neq \emptyset$. The set A will be called* maximal finitely non-Hausdorff subset of X *if A is a finitely non-Hausdorff subset of X and if B is a finitely non-Hausdorff subset of X such that $A \subset B$ then $A = B$.*

We note that in a Hausdorff space X the only maximal finitely non-Hausdorff subsets of X are the singletons.

Lemma 2.3. *Every finitely non-Hausdorff subset of a topological space X is contained in a maximal finitely non-Hausdorff subset of X.*

Proof. It is a direct corollary of Zorn's lemma. □

Now we are ready to introduce the concept of a *non-Hausdorff number* of a topological space X.

Definition 2.4. *Let X be a topological space. We define* the non-Hausdorff number $nh(X)$ of X *as follows:* $nh(X) := 1 + \sup\{|A| : A$ *is a (maximal) finitely non-Hausdorff subset of* $X\}$.

Remark 2.5. *It follows from Definition 2.4 that X is a T_2-space if and only if $nh(X) = 2$ and $2 < nh(X) \leq 1 + |X|$ whenever X is a non-Hausdorff space. Also, if X is a topological space and $A \subset X$ then $nh(A) \leq nh(X)$, and if X is an infinite set with topology generated by the open sets $\{X \setminus \{x\} : x \in X\}$ then X is a maximal finitely non-Hausdorff set and therefore $nh(X) = |X|$. Finally, using similar ideas as in Example 2.1 one can construct T_1-spaces X with one or more of the following properties:*

(i) *there exist maximal finitely non-Hausdorff subsets M and N of X such that $|M \cap N| \geq 0$ and $|M|$, $|N|$, and $|M \cap N|$ could have any cardinality which satisfy $0 \leq |M \cap N| \leq |M|$ and $0 \leq |M \cap N| \leq |N|$;*

(ii) *there exists a maximal finitely non-Hausdorff subset M and a point $x \in M$ such that $M \subsetneq \cap\{\overline{U_x} : U_x \in \mathcal{N}_x\}$ and $|M| = nh(X)$.*

We finish this section with two observations about finitely non-Hausdorff subsets of topological spaces.

Lemma 2.6. *Let X be a topological space and A be a finitely non-Hausdorff subset of X. Then $A \subset \cap\{\overline{\cap\mathcal{U}} : \mathcal{U} \in \mathcal{U}_F, \emptyset \neq F \subset A, |F| < \omega\}$.*

Proof. Let F be a nonempty subset of A, $\mathcal{U}_0 \in \mathcal{U}_F$, and $G = \cap\mathcal{U}_0$. Suppose that there exist $a_0 \in A$ such that $a_0 \notin \overline{G}$. Then there is $W_{a_0} \in \mathcal{N}_{a_0}$ such that $W_{a_0} \cap G = \emptyset$. Let $V_{a_0} = W_{a_0}$ if $a_0 \notin F$ and $V_{a_0} = U_{a_0} \cap W_{a_0}$, where $U_{a_0} \in \mathcal{U}_0$ and $U_{a_0} \in \mathcal{N}_{a_0}$, if $a_0 \in F$. Then the family $\mathcal{U}_1 := \{V_{a_0}\} \cup \{U_a : U_a \in \mathcal{U}_0, a \in F \setminus \{a_0\}\}$ has the property that $\cap\mathcal{U}_1 = \emptyset$ - a contradiction. Therefore $A \subset \overline{\cap\mathcal{U}}$ for every $\mathcal{U} \in \mathcal{U}_F$ and every nonempty subset F of A with $|F| < \omega$. Thus $A \subset \cap\{\overline{\cap\mathcal{U}} : \mathcal{U} \in \mathcal{U}_F, \emptyset \neq F \subset A, |F| < \omega\}$. □

Theorem 2.7. *Let X be a topological space and A be a maximal finitely non-Hausdorff subset of X. Then $A = \cap\{\overline{\cap\mathcal{U}} : \mathcal{U} \in \mathcal{U}_F, \emptyset \neq F \subset A, |F| < \omega\}$.*

Proof. Let A be a maximal finitely non-Hausdorff subset of X. Then it follows from Lemma 2.6 that $A \subset \cap\{\overline{\cap\mathcal{U}} : \mathcal{U} \in \mathcal{U}_F, \emptyset \neq F \subset A, |F| < \omega\}$. Suppose that there is $x_0 \in \cap\{\overline{\cap\mathcal{U}} : \mathcal{U} \in \mathcal{U}_F, \emptyset \neq F \subset A, |F| < \omega\} \setminus A$. Then $U \cap (\cap\mathcal{U}) \neq \emptyset$ for every $U \in \mathcal{N}_{x_0}$, every $\mathcal{U} \in \mathcal{U}_F$ and every nonempty finite subset F of A. Thus for the set $A_1 := A \cup \{x_0\}$ we have that if $F \subset A_1$ with $F \neq \emptyset$ and $|F| < \omega$, and $\mathcal{U} \in \mathcal{U}_F$ then $\cap\mathcal{U} \neq \emptyset$. Therefore, A_1 is a finitely non-Hausdorff subset of X and $A \subsetneq A_1$ - a contradiction with the maximality of A. □

3. Some cardinal inequalities involving the non-Hausdorff number

We begin with the generalization of Pospišil's inequalities (a), (b) and (c) for the class of all topological spaces.

Theorem 3.1. *Let X be a topological space. Then $|X| \leq 2^{2^{d(X)}} \cdot nh(X)$.*

Proof. Let D be a dense subset of X with $d(X) = |D|$ and $u = hn(X)$. For every nonempty finite subset A of X and every point $x \in A$ we choose a subset $G_{A,x}$ of D with $x \in \overline{G_{A,x}}$ in the following way:

(i) If A is a finitely non-Hausdorff subset of X then $G_{A,x} := D$; and

(ii) If A is not a finitely non-Hausdorff subset of X then for each $x \in A$ we choose $U_{A,x} \in \mathcal{N}_x$ such that $\cap\{U_{A,x} : x \in A\} = \emptyset$. Then for $x \in A$ we let $G_{A,x} := U_{A,x} \cap D$.

Now, for $x \in X$ let $\Gamma_x := \{G_{A,x} : A \subset X, \emptyset \neq |A| < \omega, x \in A\}$. Then, for each $x \in X$, Γ_x is a centered family and $\Gamma_x \in \mathcal{P}(\mathcal{P}(D))$. We claim that the mapping $x \to \Gamma_x$ from X to $\mathcal{P}(\mathcal{P}(D))$ is $(\leq u)$-to-one. Assume the contrary. Then there is a subset $K \subset X$ such that $|K| = u^+$ and every $x \in K$ corresponds to the same centered family Γ. Since $nh(X) = u$, K is not a finitely non-Hausdorff subset of X. Then there exists $F \subset K$ with $\emptyset \neq |F| < \omega$ and $\mathcal{U} \in \mathcal{U}_F$ such that $\cap\mathcal{U} = \emptyset$. Then for every $x \in F$ we have $U_{F,x} \in \Gamma$; hence Γ is not centered - a contradiction. Therefore $|X| \leq 2^{2^{|D|}} \cdot u$. □

Corollary 3.2. *Let X be a topological space. Then $w(X) \leq 2^{(2^{2^{d(X)}} \cdot nh(X))}$.*

Proof. It follows directly from Theorem 3.1 and the fact that for any topological space X, $w(X) \leq 2^{|X|}$ (see [7, 3.1a]). □

Remark 3.3. *Example 2.1 shows that the upper bound in the inequality in Theorem 3.1 is exact. To see that, let $\alpha > 2^{2^\omega}$. Then $nh(X) = \alpha$ and $\alpha = |X| \leq 2^{2^\omega} \cdot nh(X) = \alpha$.*

Theorem 3.4. *Let A be a subset of a topological space X. Then $|\overline{A}| \leq |A|^{\chi(\overline{A})} \cdot nh(\overline{A})$.*

Proof. Let $\chi(\overline{A}) = m$, $nh(\overline{A}) = u$, and $|A| = \tau$. For each $x \in \overline{A}$ let $\mathcal{V}_x$ be a local base for x in $\overline{A}$ with $|\mathcal{V}_x| \leq m$. For every $x \in \overline{A}$ and every $V \in \mathcal{V}_x$, fix a point $a_{x,V} \in V \cap A$, and let $A_x := \{a_{x,V} : V \in \mathcal{V}_x\}$. Let also $\Gamma_x := \{V \cap A_x : V \in \mathcal{V}_x\}$. Then Γ_x is a centered family. It is not difficult to see that there are at most τ^m such centered families. Indeed $A_x \in [A]^{\leq m}$ and $V \cap A_x \in [A]^{\leq m}$, for every $V \in \mathcal{V}_x$. Since $|\Gamma_x| \leq m$, each centered family Γ_x is an element of $[[A]^{\leq m}]^{\leq m}$ and therefore there are at most $(|A|^m)^m = |A|^m = \tau^m$ such families.

We claim that the mapping $x \to \Gamma_x$ is $(\leq u)$-to-one. Assume the contrary. Then there is a subset $K \subset \overline{A}$ such that $|K| = u^+$ and every $x \in K$ corresponds to the same centered family Γ. Since $nh(\overline{A}) = u$, K is not a finitely non-Hausdorff subset of $\overline{A}$. Then there exist $F \subset K$ with $\emptyset \neq |F| < \omega$ and $\mathcal{U} \in \mathcal{U}_F$ such that $\cap\mathcal{U} = \emptyset$. Hence for every $x \in F$ and $U_x \in \mathcal{U}$ we have $U_x \cap A_x \in \Gamma$; thus Γ is not centered - a contradiction.

Therefore the mapping $x \to \Gamma_x$ from $\overline{A}$ to $[[A]^{\leq m}]^{\leq m}$ is $(\leq u)$-to-one, and thus

$$|\overline{A}| \leq u \cdot (\tau^m)^m = u \cdot \tau^m \tag{1}$$

□

Corollary 3.5. *Let A be a subset of a topological space X. Then $|\overline{A}| \leq |A|^{\chi(X)} \cdot nh(X)$.*

Corollary 3.6. *Let X be a topological space. Then $|X| \leq d(X)^{\chi(X)} \cdot nh(X)$.*

Remark 3.7. *Example 2.1 shows that the upper bound in the inequality in Corollary 3.6 (and Theorem 3.1) is exact. To see that, let $\alpha > 2^\omega$. Then $nh(X) = \alpha$ and $\alpha = |X| \leq d(X)^{\chi(X)} \cdot nh(X) = \omega^\omega \cdot \alpha = \alpha$.*

The following theorem generalizes Arhangel′skiĭ's inequality for the class of T_1-topological spaces.

Theorem 3.8. *For every T_1-topological space X, $|X| \leq nh(X)^{\chi(X)L(X)}$.*

Proof. Let $\chi(X)L(X) = m$ and $nh(X) = u$. For each $x \in X$ let $\mathcal{V}_x$ be a local base for x with $|\mathcal{V}_x| \leq m$. Let x_0 be an arbitrary point in X. Recursively we construct a family $\{F_\alpha : \alpha < m^+\}$ of subsets of X with the following properties:

(i) $F_0 = \{x_0\}$ and $\overline{\cup_{\beta<\alpha} F_\beta} \subset F_\alpha$ for every $0 < \alpha < m^+$;
(ii) $|F_\alpha| \leq u^m$ for every $\alpha < m^+$;
(iii) for every $\alpha < m^+$, and every $F \subset \overline{\cup_{\beta<\alpha} F_\beta}$ with $|F| \leq m$ if $X \setminus \cup\mathcal{U} \neq \emptyset$ for some $\mathcal{U} \in \mathcal{U}_F$, then $F_\alpha \setminus \cup\mathcal{U} \neq \emptyset$.

Suppose that the sets $\{F_\beta : \beta < \alpha\}$ satisfying (i)-(iii) have already been defined. We will define F_α. Since $|F_\beta| \leq u^m$ for each $\beta < \alpha$, we have $|\cup_{\beta<\alpha} F_\beta| \leq u^m \cdot m^+ = u^m$. Then it follows from Corollary 3.5, that $|\overline{\cup_{\beta<\alpha} F_\alpha}| \leq u^m$. Therefore there are at most u^m subsets F of $\overline{\cup_{\beta<\alpha} F_\alpha}$ with $|F| \leq m$ and for each such set F we have $|\mathcal{U}_F| \leq m^m = 2^m \leq u^m$. For each $F \subset \overline{\cup_{\beta<\alpha} F_\alpha}$ with $|F| \leq m$ and each $\mathcal{U} \in \mathcal{U}_F$ for which $X \setminus \cup\mathcal{U} \neq \emptyset$ we choose a point in $X \setminus \cup\mathcal{U} \neq \emptyset$ and let E_α be the set of all these points. Clearly $|E_\alpha| \leq u^m$. Let $F_\alpha = \overline{E_\alpha \cup (\cup_{\beta<\alpha} F_\alpha)}$. Then it follows from our construction that F_α satisfies (i) and (iii) while (ii) follows from Corollary 3.5.

Now let $G = \cup_{\alpha<m^+} F_\alpha$. Clearly $|G| \leq u^m \cdot m^+ = u^m$. We will show that G is closed. Suppose the contrary and let $x \in \overline{G} \setminus G$. Then for each $U \in \mathcal{V}_x$ we have $U \cap G \neq \emptyset$ and therefore there is $\alpha_U < m^+$ such that $U \cap F_{\alpha_U} \neq \emptyset$. Since $|\{\alpha_U : U \in \mathcal{V}_x\}| \leq m$, there is $\beta < m^+$ such that $\beta > \alpha_U$ for every $U \in \mathcal{V}_x$ and therefore $x \in F_\beta \subset G$ - a contradiction.

To finish the proof it remains to check that $G = X$. Suppose that there is $x \in X \setminus G$. Since X is T_1, for every $y \in G$ there is $V_y \in \mathcal{V}_y$ such that

$x \notin V_y$. Then $\{X \setminus G\} \cup \{V_y : y \in G\}$ is an open cover of X. Thus there exists $F \subset G$ with $|F| \leq m$ such that $G \subset \cup_{y \in F} V_y$. Since $|F| \leq m$, there is $\beta < m^+$ such that $F \subset F_\beta$. Then for $\mathcal{U} := \{V_y : y \in F\}$ we have $\mathcal{U} \in \mathcal{U}_F$ and $x \in X \setminus \cup\mathcal{U}$. Therefore $F_{\beta+1} \setminus \cup\mathcal{U} \neq \emptyset$ - a contradiction. □

Corollary 3.9. *Let X be an infinite T_1-topological space with $nh(X)$ not greater than the cardinality of the continuum. Then $|X| \leq 2^{\chi(X)L(X)}$.*

Proof. For every infinite T_1-topological space X either $\chi(x)$ or $L(X)$ is infinite. □

Example 3.10. *Let I be the unit interval with its standard topology and Y be a set with cardinality $\alpha > 2^\omega$. Let $X := I \cup Y$ be the topological space with the following topology: every point $y \in Y$ is an open set but not a closed set in X and $U \subset X$ is a neighborhood of a point $i \in I$ if and only if $U = V \cup Y$ where $V \subset I$ is a standard neighborhood of i in I. Then X is a T_0 topological space with $\chi(X) = L(X) = \omega$ and $nh(X) = 2^\omega$ but $|X| = \alpha > (2^\omega)^{\omega \cdot \omega} = 2^\omega$. Therefore Theorem 3.8 is not necessarily valid for T_0-topological spaces.*

Remark 3.11. *For results about cardinal inequalities for topological spaces involving the Hausdorff number of a topological space see* [2]*; for results involving the Urysohn number of a space see* [4], [3] *and* [5]*; and for results involving the non-Urysohn number of a topological space see* [6].

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