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# TOPOLOGICAL SPACES WITH NO COMPACTLY DETERMINED EXTENSIONS

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The $\mathcal{P}$ spaces with no compactly determined $\mathcal{P}$- extensions are characterized for $\mathcal{P}=LM_2$, $T_2$ and $T_3$. It is proved that every $LM_2$ closed space is compact. Examples of absolutely $LM_2$- closed spaces which are not compact are given.

An extension $Y$ of a topological space $X$ is caled a compactly determined extension if for every point $y\in Y\setminus X$ there is $K\subset X$ such that $y\in\overline{K}^Y$ and $\overline{K}^Y$ is compact [3]. All compactly determined $\mathcal{P}$- extensions of a $\mathcal{P}$ space for $\mathcal{P}=T_3$ and $\mathcal{P}=T_2$ were characterized by D.Doithinov [3] and I.Gotchev [6] in terms of supertopologies. In this paper the $\mathcal{P}$- spaces with no compactly determined $\mathcal{P}$- extensions are characterized for $\mathcal{P}=LM_2$, $T_2$ and $T_3$. A space $X$ is an $LM_2$ space if every compact subspace of $X$ is $T_2$ [7]. Obviously $T_2\subset LM_2\subset T_1$. For $\mathcal{P}=LM_2$ the spaces with no compactly determined $LM_2$-extensions coincide with the absolutely $LM_2$- closed spaces. This fact was observed by D.Dikranjan and E.Guili. A $\mathcal{P}$ space $X$ is absolutely $\mathcal{P}$- closed if for every space $Z\in\mathcal{P}$ and for every embedding $h:X\to Z$ there exist a space $Y\in\mathcal{P}$ and two continuous maps $f,g:Z\to Y$ such that $h(X)=\{z\in Z\mid f(z)=g(z)\}$ (see [1], [2] and [5]). A $\mathcal{P}$- space $X$ is $\mathcal{P}$- closed if $X$ is a closed subspace in every $\mathcal{P}$- space in which it is embedded. In this paper also we will show that every $LM_2$ lcsed space is compact. Examples of absolutely $LM_2$-closed spaces which are not compact are given, which answer a question of D.Dikranjan and E.Guili.

In this paper every topological space is $T_1$ and every proximity $\delta$ in a topological space $X$ will be compatible with the topology on $X$. (For the definition of proximity see [4].) Let us recal that in a space $X$ there exist a proximity $\delta$ if and only if $X$ is a completely regular space.

Let $X$ be a topological space, a subset $A$ of $X$ is $\theta$ - closed (see [8]) if for every point $x\in X\setminus A$ there is a neighbourhood $U$ of $x$ such that $\overline{U}\cap A=\emptyset$ and $A$ is $\theta'$- closed if for every closed set $F\subset X\setminus A$ there is an open set $U$ of $X$ such that $F\subset U$ and $\overline{U}\cap A=\emptyset$.

Obviously, every $\theta'$- closed subset of $X$ is $\theta$- closed and every $\theta$ closed subset of $X$ is closed. If $X$ is regular then every closed set in $X$ is $\theta$ - closed and if $X$ is normal then every closed set in $X$ is $\theta'$- closed.

Let $A\subset X$ and $\delta$ be a proximity in $A$, we say that $A$ is $\delta$- placed in $X$ if for every pair $B,C\subset A$ such that $B\bar{\delta}C$ there are disjoint open sets $U$ and $V$ in $X$ with $B\subset U$ and $C\subset V$.

It is clear that if $A$ is $\delta$- placed in $X$ then $A$ is a completely regular space and if $X$ is a Hausdorff space then every compact subset $B$ of $X$ is

$\theta$ -closed and $\delta$ -placed in $X$, where $\delta$ is the standared proximity in $B$ (see [4]).

A subset $A$ of $X$ is compactly related to $X$ is the elements of every filter of closed sets on $\overline{X\setminus A}$ which has empty intersection, intersect $A$.

Let $X$ and $Y$ be topological spaces, $A$ be a closed set in $X$ and $f: A \to Y$ be a mapping. Then the adjunction space $X \cup_f Y$ (see 4) will be denoted by $X \cup_A Y$ in the case $A \subset Y$ and $f: A \to Y$ is the identical embedding.

Lemma 1. Let $A$ be a closed set in $X$ and $\delta A$ be a $T_1$ - compactification of $A$. Then $X$ is compactly related to $A$ if and only if $X \cup_A \delta A$ is compact.

Proof. Let $X$ be compactly related to $A$ and let $Y = X \cup_A \delta A$. We prove that $Y$ is compact. Let $\mathcal{F}$ be an ultrafilter of closed sets on $Y$. If there exists $F \in \mathcal{F}$ such that $F \cap X = \emptyset$ then $F \subset \delta A$. Moreover $A$ is a closed set in $X$, hence $\delta A$ is a closed set in $Y$. Since $\delta A$ is compact and $F \subset \delta A$ then $\mathcal{F}$ has a cluster point in $\delta A$ and so in $Y$. Let now for every $F \in \mathcal{F}$, $F \cap X \neq \emptyset$. Then $\mathcal{F}_1 = \{F \cap X \mid F \in \mathcal{F}\}$ is an ultrafilter of closed sets in $X$. If there exists $F_1 \in \mathcal{F}_1$ such that $F_1 \cap \overline{X\setminus A}^X = \emptyset$ then $F_1 \subset A \subset \delta A$ and therefore $\mathcal{F}_1$ has cluster points in $\delta A$ and hence $\mathcal{F}$ has cluster points in $Y$. If every $F_1 \in \mathcal{F}_1$ meets $\overline{X\setminus A}^X \neq \emptyset$ then $\mathcal{F}_2 = \{F_1 \cap \overline{X\setminus A}^X \mid F_1 \in \mathcal{F}_1\}$ is an ultrafilter of closed sets in $\overline{X\setminus A}^X$. Suppose there exists $F_2 \in \mathcal{F}_2$ such that $F_2 \cap A = \emptyset$. Since $X$ is a compactly related to $A$ then $\mathcal{F}_2$ and consequently also $\mathcal{F}$ has cluster points in $Y$. If for every $F_2 \in \mathcal{F}_2, F_2 \cap A \neq \emptyset$ then $\mathcal{F}_3 = \{F_2 \cap A \mid F_2 \in \mathcal{F}_2\}$ is an ultrafilter of closed sets in $A$. Then $\mathcal{F}_3$ and also $\mathcal{F}$ has cluster points in $\delta A$ and hence in $Y$. Therefore $Y$ is compact.

Now let $Y = X \cup_A \delta A$ be compact. We will prove that is compactly related to $A$. Let $\mathcal{F}$ be an ultrafilter of closed sets in $\overline{X\setminus A}^X$ and assume that $\mathcal{F}$ has no cluster points. The space $Y$ is compact, hence the filter $\mathcal{F}' = \{\overline{F}^Y \mid F \in \mathcal{F}\}$ has a cluster point $x$ in $Y$. Since $x$ is not a cluster point of $\mathcal{F}$ then $x \notin X$. Hence $x \in Y \setminus X$.

Let us assume that there exists a closed set $F \in \mathcal{F}$ such that $F \cap A = \emptyset$. Then $X \setminus F$ is an open set in $X$ and $F \subset X \setminus A$. Thus $\delta A \cup (X \setminus F)$ is an open neighbourhood of $x$ in $Y$ and $F \cap (\delta A \cup (X \setminus F)) = \emptyset$. Therefore $x \notin \overline{F}^Y$ and hence $x$ is not a cluster point of $\mathcal{F}'$ a contradiction. So for every $F \in \mathcal{F}$ we have $F \cap A \neq \emptyset$. This means that $X$ is compactly related to $A$.

Corollary. Let $A$ be a compact subspace of $X$. Then $X$ is compactly related to $A$ if and only if $X$ is compact.

Lemma 2. Let $X$ be a Hausdorff space. $A$ be a closed set of $X$, $\delta$ be a proximity in $A$ and $\delta A$ be the compactification of $A$ related to $\delta$ (see [4]). Then $X \cup_A \delta A$ is a Hausdorff space if and only if $A$ is $\theta$ - closed and $\delta$ placed in $X$.

Proof. Let $Y = X \cup_A \delta A$ be a Hausdorff space. We shall prove that $A$ is $\delta$ placed in $X$. Let $B, C \subset A$ and $B \overline{\delta} C$. It is easily seen that $B \delta C$ if and only if $B \cap C = \emptyset$. The space $\delta A$ is Hausdorff and compact, so there exist open sets $U$ and $V$ in $Y$ such that $\overline{B}^Y \subset U, \overline{C}^Y \subset V$ and $U \cap V = \emptyset$. So if $U' = U \cap X$ and $V' = V \cap X$ then $A \subset U', C \subset V'$ and $U' \cap V' = \emptyset$. Thus $A$ is $\delta$- placed in $X$. Let now $x \in Y \setminus \delta A$. Since $\delta A$ is compact and $Y$ is a Hausdorff space, there exist open sets $U$ and $V$ in $Y$ such $x \in U$. $\delta A \subset V$ and $U \cap V = \emptyset$. Thus $\overline{U}^Y \cap \delta A = \emptyset$ and hence $\overline{U}^Y \cap A = \emptyset$.

Since $\overline{U}^Y \cap X = \overline{U}^Y$ then $\overline{U}^Y$ is a closed set in $X$. Therefore $A$ is $\theta$ closed in $X$.

Now let $A$ be $\theta$- closed and $\delta$- placed in $X$. We will prove that $Y = X \cup_A \delta A$ is a Hausdorff space. Let $x_1 \in X \setminus A = Y \setminus \delta A$ and $x_2 \in X \setminus A$. The space $X \setminus A$ is open in $Y$ and $X$ is a Hausdorff space, so there exist disjoint open sets $\mathcal{U}$ and $V$ in $Y$ such that $x_1 \in \mathcal{U}$ and $x_2 \in V$.

Now let $x_2 \in \delta A$. Since $A$ is a $\theta$-closed set in $X$ then there exist open sets $\mathcal{U}$ and $V$ in $X$ such that $x_1 \in \mathcal{U}, A \subset V$ and $\mathcal{U} \cap V = \phi$. Hence if $W = V \cup \delta A$ then $W$ is an open neighbourhood of $x_2$ in $Y$. $\mathcal{U}$ is an open neighbourhood of $x_1$ in $Y$ and $\mathcal{U} \cap W = \phi$. Let now $, x_2 \in \delta A$. Since $\delta A$ is a closed Hausdorff subspace in $Y$, there exist open sets $\mathcal{U}_1$ and $\mathcal{U}_2$ in $\delta A$ such that $\overline{\mathcal{U}_1}^Y \cap \overline{\mathcal{U}_2}^Y = \phi$. Let $F_1 = \overline{\mathcal{U}_1}^Y \cap A = \overline{\mathcal{U}_1}^Y \cap X$ and $F_2 = \overline{\mathcal{U}_2}^Y \cap A = \overline{\mathcal{U}_2}^Y \cap X$. Clearly $F_1 \bar{\delta} F_2$. Since $A$ is $\delta$-placed in $X$ then there exist open sets $\mathcal{U}_1'$ and $\mathcal{U}_2'$ in $X$ such that $F_1 \subset \mathcal{U}_1'$, $F_2 \subset \mathcal{U}_2'$ and $\mathcal{U}_1' \cap \mathcal{U}_2' = \phi$. On the other hand there exist open sets $\mathcal{U}_1''$ and $\mathcal{U}_2''$ in $X$ such that $\mathcal{U}_1'' \cap A = \mathcal{U}_1 \cap A$ and $\mathcal{U}_2'' \cap A = \mathcal{U}_2 \cap A$. Thus if $V_1 = \mathcal{U}_1 \cup (\mathcal{U}_1' \cap \mathcal{U}_1'')$ and $V_2 = \mathcal{U}_2 \cup (\mathcal{U}_2' \cap \mathcal{U}_2'')$ then $V_1$ and $V_2$ are disjoint open sets in $Y$ such that $x_1 \in V_1$ and $x_2 \in V_2$. Therefore $Y$ is a Hausdorff space.

The following lemma could be proved in the same way as the above lemma.

Lemma 3. Let $X$ be a regular space, $A$ be a closed set in $X$, $\delta$ be a proximity in $A$ and $\delta A$ be the compactification of $A$, related to $\delta$.

Then $X \cup_A \delta A$ is a regular space if and only if $A$ is $\theta$-closed and $\delta$ placed in $X$.

Theorem 1. Let $X$ be an $LM_2$-space. Then $X$ has no compacity determined $LM_2$ extension if and only if every subset $A$ of $X$ which satisfies the following two conditions is compact.

i) Every compactly related to $A$ subset of $X$ is a Hausdorff space.

ii) There exist a proximity $\delta$ in $A$ such that $A$ is $\theta$-closed and $\delta$ placed in every compactly related to $A$ subset of $X$.

Proof. Let us assume that there exists compactly determined $LM_2$-extension $Y$ of $X$ and $Y \neq X$. We may assume without loss of generality that there exists a closed set $A$ in $X$ such that $\overline{A}^Y$ is compact and $Y = X \cup \overline{A}^Y$. We shall prove that $A$ satisfies the conditions i) and ii). Let $Z \subset X$ be compactly related to $A$. Then $A \subset Z$ and $A$ will be a closed set in $Z$. Furthermore $Z \cup_A \overline{A}^Y$ will be homeomorphic to $Z \cup \overline{A}^Y$. By Lemma 1 we know that $Z \cup_A \overline{A}^Y$ is compact. Thus $Z \cup \overline{A}^Y$ is a compact subspace of $Y$. Since $Y$ is $LM_2$ then $Z \cup \overline{A}^Y$ is a Hausdorff space and hence $Z$ is a Hausdorff space. If $\delta$ is the proximity of $A$ induced by the standard proximity $\delta'$ of the compact space $\overline{A}^Y$ then $A$ will be $\theta$-closed and $\delta$ placed in $Z$ by Lemma 2. Therefore $A$ satisfies the conditions i) and ii) but $A$ is not compact - a contradiction.

Now assume that there exists $A \subset X$ such that the conditions i) and ii) are satisfied and $A$ is not a compact space. Since $A$ is compactly related to $A$ ii) yields that $A$ is a compactly regular space. Furthermore if $x \in X$ then $A \cup \{x\}$ is compactly related to $A$ and by ii) $A$ is $\theta$-closed in $A \cup \{x\}$. Hence $A$ is a closed set in $X$. Let $\delta$ be the proximity in $A$ which satisfies ii) and let $\delta A$ be the compactification of $A$ related to $\delta$. Set $Y = X \cup_A \delta A$, clearly $Y$ is a compactly determined extension of $X$. We will prove that $Y$ is an $LM_2$-space. Let $Z' \subset Y$, be a compact subspace. Without loss of generality we may assume that $\delta A \subset Z'$. For $Z = Z' \cap X$ the space $Z \cup_A \delta A$ is homeomorphic to $Z'$. By Lemma 1 and by the compactness of $Z \cup_A \delta A$ it follows that $Z$ is compactly related to $A$. But by i) $Z$ is a Hausdorff space and by Lemma 2 $Z'$ is a Hausdorff space.

The above result is in fact a chatacterization of the absolutely $LM_2$-closed spaces.

Theorem 2. A Hausdorff space $X$ has no Hausdorff compactly determined extensions if and only if every $\theta$-closed and $\delta$-placed subset of $X$ is compact.

Proof. Follows by Lemma 2.

Theorem 3. A regular space $X$ has no regular compactly determined extensions if and only if every $\theta$- closed and $\delta$- placed subset of $X$ is compact.

Proof. Follows by Lemma 3.

Now we will show that every $LM_2$-closed space is compact.

Lemma 4. Let $x$ be a non isolated point for a topological space $X$. If $x$ has no compact neighbourhood in $X$ then $X$ is not an $LM_2$ - closed space.

Proof. Let us assume that for some non isolated point $x \in X$ there is no open set $\mathcal{U}$ in $X$ such that $x \in \mathcal{U}$ and $\overline{\mathcal{U}}$ is compact. We consider the space $Y = X \cup \{y\}$ where $y \notin X$ with the following topology: $\{y\}$ is a closed set and for a base of neighbourhoods of $\{y\}$ we take the family $\{\mathcal{U} \mid \mathcal{U} = (V \setminus F) \cup \{y\}\}$, where $V$ is an open set in $X$, $x \in V$ and $F$ is a compact set in $X$. It is easily seen that it is a topology on $Y$. We will prove that $Y$ is an $LM_2$ space. Let $A \subset Y$ and $A$ be a compact subspace. If $y \notin A$ then $A \subset X$ and $A$ is Hausdorff compact. Now let $y \in A$, then $\{x\} \cup A$ is compact. If $\mathcal{V}_x$ and $\mathcal{V}_y$ are the filters of neighbourhoods of $x$ and $y$ on $Y$ then for every $\mathcal{U} \in \mathcal{V}_x$ we have $\mathcal{U} \cup \{y\} \in \mathcal{V}_y$. Thus $F = (\{x\} \cup A) \setminus \{y\}$ is a compact space in $X$ and hence $F$ is a closed set in $X$. Thus there exists a neighbourhood $\mathcal{U}$ of $x$ in $X$ such that $\mathcal{U} \setminus F \neq \emptyset$. Therefore $W = \{y\} \cup (\mathcal{U} \setminus F)$ is a neighbourhood of $y$ in $Y$ avoiding $F$, hence $y$ is an isolated point in $A$. Thus $Y$ is an $LM_2$ - extension of $X$, then $X$ is not an $LM_2$ closed space.

Lemma 5. Let $X$ be an $LM_2$-space. If every point in $X$ has a compact neighbourhood in $X$ then $X$ is a Hausdorff space.

Proof. It is obvious.

Theorem 4. An $LM_2$-space $X$ is $LM_2$-closed if and only if $X$ is a compact Hausdorff space.

Proof. It is obvious that if $X$ is a compact Hausdorff space then $X$ is an $LM_2$-closed space. Now let $X$ be an $LM_2$-closed space. By Lemma 4 it follows that every point $x \in X$ has an open neighbourhood with compact closure. Thus by Lemma 5 $X$ is a locally compact Hausdorff space. Therefore $X$ is a completely regular space and by the $LM_2$ closedness, $X$ is a compact.

The following example shows that there exists an absolutely $LM_2$-closed space which is not compact.

Example. Let $X = [0,1]$ and $\mathcal{T}$ be the usual topology on $X$. Let $\mathcal{T}_1$ be the coarsest topology on $X$ such that $\mathcal{T} \subset \mathcal{T}_1$ and $\{\frac{1}{n}\}_{n=1}^{\infty}$ is a closed set in $\mathcal{T}_1$. Then $(X, \mathcal{T}_1)$ is $LM_2$- closed but it is not even countably compact.

Proof. The absolute $LM_2$- closedness of $(X, \mathcal{T}_1)$ may be proved by means of Theorem 1. Obviously $X$ is not countably compact.

Other examples of $LM_2$ spaces which have no compactly determined extensions and which are not Hausdorff compact may be obtained from the following result which we give without proof.

Theorem 5. Let $X$ be a first countable compact Hausdorff space and there are two dense, disjoint sets $D_1$ and $D_2$ such that $D_1 \cup D_2 = X$. Let $\mathcal{T}_1$ be the coarsest topology on $X$ such that $\mathcal{T} \subset \mathcal{T}_1$ and $D_2 \in \mathcal{T}_1$. Then $(X, \mathcal{T}_1)$ is absolutely $LM_2$ - closed.

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## REFERENCES

1. D.Dikranjan, E.Guili. Urysohn closed spaces-old and new (manuscript).
2. D.Dikranjan, E.Guili. S(n) - -closed spaces (to appear in Topology 1987)
3. D.Doitchinov. Compactly determined extensions of topological spaces. SERDICA, Bulg. math. Publ., 11, 1985, 269-286.
4. R.Engelking. General Topology. Warszawa, 1977.
5. E.Guili, M.Husek. A diagonal theorem for epireflective subcategories of Top and co-well-poweredness, Annali Mat. (to appear).
6. I.Gotchev. Superneighbourhood spaces and extensions of topological spaces. (to appear in Ann. de l Univ. de Sofia, Fac. de Math. et Mec.)
7. R-E Hoffman. On weak Hausdorff spaces. Arch. Math. 32(1979), 487-504.
8. H.Velicko. H-closed topological spaces. Mat.Sb. (N.S.) 70 (112) (1°66) 98-112. Amer. Math. Soc. Transl. 78(2) (1969) 103-118.


## ТОПОЛОГИЧНИ ПРОСТРАНСТВА, НЕПРИТЕЖАВАЩИ КОМПАКТНО ОПРЕДЕЛЕНИ РАЗШИРЕНИЯ


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Характеризирани са $\mathcal{P}$ пространствата, непритежаващи компактно определени разширения за $\mathcal{P} = LM_2.T_2$ и $T_3$. Доказано е, че всяко $LM_2$- затворено пространство е компактно. Посочени са примери на абсолютно $LM_2$-затворени пространства, които не са компактни.