1. SET THEORY

1. SETS AND SUBSETS. BASIC SET OPERATIONS

Intuitively, a set is any well-defined list, collection, or class of objects. These objects are called the elements or members of the set.

**Example:** \( A = \{1,2,3,4\} \), \( B = \{x \mid x \text{ is even natural number}\} \).

In the previous example the set \( A \) is written in so called tabular form while the set \( B \) is written in set-builder form. Notice that the vertical line “\( | \)” in \( B \) is read “such that”.

If an object \( x \) is a member of a set \( A \), i.e. \( A \) contains \( x \) as one of its elements, then we write \( x \in A \), which can also be read “\( x \) belongs to \( A \)”. If, on the other hand, an object \( x \) is not a member of a set \( A \), i.e. \( A \) does not contain \( x \) as one of its elements, then we write \( x \notin A \).

Set \( A \) is equal to set \( B \) if they both have the same members. We denote the equality of sets \( A \) and \( B \) by \( A = B \).

The empty set is a set which contains no elements. This set is sometimes called the null set and we denote it by the symbol \( \emptyset \).

If every element in a set \( A \) is also a member of a set \( B \), then \( A \) is called a subset of \( B \). We denote this by \( A \subseteq B \).

The family of all the subsets of any set \( A \) is called the power set of \( A \) and is denoted by \( P(A) \) or by \( 2^{A} \).

The union of sets \( A \) and \( B \) is the set of all elements which belong to \( A \) or to \( B \) or to both. We denote the union of \( A \) and \( B \) by \( A \cup B \) which is usually read “\( A \) union \( B \)”. The union of \( A \) and \( B \) may also be defined concisely by \( A \cup B = \{x \mid x \in A \text{ or } x \in B\} \).

**Remark:** \( A \cup B = B \cup A \).

The intersection of sets \( A \) and \( B \) is the set of elements which are common to \( A \) and \( B \). We denote the intersection of \( A \) and \( B \) by \( A \cap B \) which is read “\( A \) intersection \( B \)”. The intersection of \( A \) and \( B \) may also be defined concisely by \( A \cap B = \{x \mid x \in A \text{ and } x \in B\} \).

**Remark:** \( A \cap B = B \cap A \).

If sets \( A \) and \( B \) have no elements in common, then we say that \( A \) and \( B \) are disjoint and we denote that by \( A \cap B = \emptyset \).

The difference of sets \( A \) and \( B \) is the set of elements which belong to \( A \) but which do not belong to \( B \). We denote the difference of \( A \) and \( B \) by \( A \setminus B \) which is read “\( A \) difference \( B \)” or, simply, “\( A \) minus \( B \)”.

The difference of \( A \) and \( B \) may also be defined concisely by \( A \setminus B = \{x \mid x \in A \text{ and } x \notin B\} \).

In any applications of the theory of sets, all the sets under investigation will likely be subsets of a fixed set. We call this set the universal set and denote it by \( S \).

The complement of a set \( A \) is the set of elements which do not belong to \( A \), that is, the difference of the universal set \( S \) and \( A \). We denote the complement of \( A \) by \( A' \). The complement of \( A \) may also be defined concisely by \( A' = \{x \mid x \in S \text{ and } x \notin A\} \).

**Remark:** \( A \cup A' = S \), \( A \cap A' = \emptyset \), \( (A')' = A \), \( S' = \emptyset \), \( \emptyset' = S \).

**Theorem 1.** \( A \subseteq B \) implies \( A \cap B = A \), \( A \cup B = B \), \( B' \subseteq A' \), \( A \cup (B \setminus A) = B \).

**Theorem 2.** (Algebra of sets)

a) Idempotent Laws: \( A \cup A = A \), \( A \cap A = A \).
b) Associative Laws: \((A \cup B) \cup C = A \cup (B \cup C)\), \((A \cap B) \cap C = A \cap (B \cap C)\).

c) Commutative Laws: \(A \cup B = B \cup A\), \(A \cap B = B \cap A\).

d) Distributive Laws: \(A \cup (B \cap C) = (A \cup B) \cap (A \cup C)\), \(A \cap (B \cup C) = (A \cap B) \cup (A \cap C)\).

e) Identity Laws: \(A \cup \emptyset = A\), \(A \cap S = A\), \(A \cup S = S\), \(A \cap \emptyset = \emptyset\).

f) Complement Laws: \(A \cup A' = S\), \(A \cap A' = \emptyset\), \((A')' = A\), \(S' = \emptyset\), \(\emptyset' = S\).

g) De Morgan’s Laws: \((A \cup B)' = A' \cap B'\), \((A \cap B)' = A' \cup B'\).

### 2. RELATIONS

Let \(A\) and \(B\) be sets. Any subset \(R^* \subseteq A \times B\) defines a relation \(R\) from \(A\) to \(B\). If \((x, y) \in R^*\) we write \(x R y\) which reads “\(x\) is related to \(y\)”. If \(A = B\) then we say that \(R\) is a relation in \(A\).

A function \(f : A \to B\) is a subset of \(A \times B\) in which each \(a \in A\) appears in one and only one ordered pair belonging to \(f\). Since every subset of \(A \times B\) is a relation, a function is a special type of a relation. A function \(f : A \to B\) is said to be injective if \(f(a_1) = f(a_2)\) implies \(a_1 = a_2\), surjective if \(f(A) = B\), and bijective if it is both injective and surjective.

Let \(R\) be a relation in \(A\) and \(x, y \in A\).

a) \(R\) is called a reflexive relation if \(x R x\) for \(\forall x \in A\);

b) \(R\) is called a nonreflexive relation if \(-\exists x \in A : x R x\);

c) \(R\) is called a symmetric relation if \(x R y \Rightarrow y R x\) for \(\forall x, y \in A\);

d) \(R\) is called an anti-symmetric relation if \(x R y \text{ and } x R y \Rightarrow x = y\);

e) \(R\) is called a transitive relation if \(x R y \text{ and } y R z \Rightarrow x R z\);

f) \(R\) is called an equivalence relation if \(R\) is reflexive, symmetric, and transitive.

Let \(\{C_i\}_{i \in I}\) be a family of non-empty subsets of \(A\). Then \(\{C_i\}_{i \in I}\) is called a partition of \(A\) if \(\bigcup_{i \in I} C_i = A\) and \(C_i \cap C_j = \emptyset\) for every \(i \neq j\). Each \(C_i\) is called an equivalence class of \(A\).

**Proposition 1:** Let \(R\) be an equivalence relation in a set \(A\) and, for every \(\alpha \in A\), let \(C_\alpha = \{x | (x, \alpha) \in R\}\). Then the family of sets \(\{C_\alpha\}_{\alpha \in A}\) is a partition of \(A\).

The set \(C_\alpha\) is called the equivalence class determined by \(\alpha\), and the set of equivalence classes \(\{C_\alpha\}_{\alpha \in A}\) is denoted by \(A / R\) and called the quotient set.

**Proposition 2:** Let \(\{C_i\}_{i \in I}\) be a partition of \(A\) and let \(R\) be the relation in \(A\) defined in the following way: \(x R y \iff x, y \in C_i\) for some \(i \in I\). Then \(R\) is an equivalence relation in \(A\).

### 3. ORDERINGS

Let \(A\) be a set, \(B \subseteq A\), and \(x, y, z, a, b\) be elements of \(A\) (or \(B\)).

A partial order in \(A\) is a relation \(R\) in \(A\) which is transitive: \(x R y \text{ and } y R z \Rightarrow x R z\). We denote \(x R y\) by \(x < y\) if \(R\) is also nonreflexive (\(-\exists x : x R x\)) or by \(x \leq y\) if \(R\) is also reflexive (\(x R x\) for \(\forall x\)) and anti-symmetric (\(x R y \text{ and } x R y \Rightarrow x = y\)).

A set \(A\) together with a specific partial order relation \(R\) in \(A\) is called a partially ordered set and is denoted by \((A, R)\).
(A, <) is linearly (totally) ordered if \( x < y, x = y, y < x \) for \( \forall x, y \in A \). (A, \( \leq \)) is linearly (totally) ordered if \( x \leq y \text{ or } y \leq x \) for \( \forall x, y \in A \).

\( a \) is a first element in \((A, <)\) if \( a < x \) for \( \forall x \neq a \). \( a \) is a first element in \((A, \leq)\) if \( a \leq x \) for \( \forall x \).

\( b \) is a last element in \((A, <)\) if \( x < b \) for \( \forall x \neq b \). \( b \) is a last element in \((A, \leq)\) if \( x \leq b \) for \( \forall x \).

**Remark:** A linearly ordered set can have at most one first and one last element.

\( a \) is a maximal element in \((A, <)\) if \( \neg \exists x : a < x \). \( a \) is a maximal element in \((A, \leq)\) if from \( a \leq x \) follows \( a = x \).

\( b \) is a minimal element in \((A, <)\) if \( \neg \exists x : x < b \). \( b \) is a minimal element in \((A, \leq)\) if from \( x \leq b \) follows \( b = x \).

\( a \in (A, <) \) is a lower bound of \( B \) if \( \neg \exists x \in B : x < a \). \( a \in (A, \leq) \) is a lower bound of \( B \) if \( a \leq x \) for \( \forall x \in B \).

\( b \in (A, <) \) is an upper bound of \( B \) if \( \neg \exists x \in B : b < x \). \( b \in (A, \leq) \) is an upper bound of \( B \) if \( x \leq b \) for \( \forall x \in B \).

If \( a \in (A, R) \) is a lower bound of \( B \) and \( yRa \) for every other lower bound \( y \) of \( B \), then \( a \) is called the greatest lower bound (g.l.b.) or infimum of \( B \) and is denoted by \( \inf(B) \).

If \( a \in (A, <) \) is an upper bound of \( B \) and \( aRy \) for every other upper bound \( y \) of \( B \), then \( a \) is called the least upper bound (l.u.b.) or supremum of \( B \) and is denoted by \( \sup(B) \).

\((A, <, A)\) and \((B, <, B)\) have the same order type (or are similar) if there exists a bijective function \( f : A \to B \) such that if \( a_1 < A a_2 \) then \( f(a_1) < B f(a_2) \).

Let \( A \) be a linearly ordered set with the property that every nonempty subset of \( A \) contains a first element. Then \( A \) is called a well-ordered set.

Let \( A \) be a well-ordered set and \( \alpha \in A \). The set \( S_\alpha = \{ \beta \mid \beta \in A, \beta < \alpha \} \) is called the initial segment of \( \alpha \).

**4. AXIOMS**

**Principle of Mathematical Induction.**

Let \( P \) be a subset of the set \( N \) of the natural numbers with the following properties:

a) \( 1 \in P \);

b) \( n \in P \) implies \( n + 1 \in P \).

Then \( P = N \).

**Principle of Transfinite Induction.**

Let \( P \) be a subset of a well-ordered set \( A \), \( \alpha_0 \) be the first element of \( A \), and \( A \) has the following properties:

a) \( \alpha_0 \in P \);

b) \( S_\alpha \subseteq P \) implies \( \alpha \in P \).

Then \( P = A \).

**The Hausdorff Maximal Principle.**

Every partially ordered set has a maximal linearly ordered subset.
Zorn’s Lemma.  
If $X$ is a partially ordered set and every linearly ordered subset of $X$ has an upper bound, then $X$ has a maximal element.

The Well Ordering Principle.  
Every nonempty set $X$ can be well ordered.

The Axiom of Choice.  
If $\{X_a\}_{a \in A}$ is a nonempty collection of nonempty sets, then $\prod_{a \in A} X_a$ is nonempty.

5. CARDINAL NUMBERS

Set $A$ is equivalent (equipollent) to set $B$, denoted by $A \sim B$ if there exists a bijective function $f : A \to B$. (Cantor)

**Theorem 1:** The relation in sets defined by $A \sim B$ is an equivalence relation.

Let $A$ be any set and let $\mathbb{N}$ denote the family of sets, which are equivalent to $A$. Then $\mathbb{N}$ is called a cardinal number and is denoted by $\mathbb{N} = \#(A)$, $\mathbb{N} = \text{card}(A)$, or $\mathbb{N} = |A|$.

A set is infinite if it is equivalent to a proper subset of itself. Otherwise, a set is finite. The cardinal number of each of the sets $\emptyset$, $\{1\}$, $\{1,2\}$, ... is denoted by 0, 1, 2, 3, ... respectively, and is called a finite cardinal.

A set is called countable if it is finite or equivalent to the set $\mathbb{N}$ of the natural numbers.

Let a set $A$ be equivalent to the $\mathbb{N}$ of the natural numbers. Then $A$ is said to have cardinality $\mathbb{N}_0$.

**Theorem 2:** Every infinite set contains a subset, which is countable.

**Theorem 3:** A subset of a countable set is countable.

**Theorem 4:** Let $\{A_i\}_{i \in I}$ be a countable family of countable sets. Then $\bigcup_{i \in I} A_i$ is countable.

**Examples:** $\mathbb{Z}$, $\mathbb{N}^2$, $\mathbb{N}^n$, $\mathbb{Q}$, and the set of points in the plane with rational coordinates are countable sets.

**Theorem 5:** The unit interval $[0,1]$ is not a countable set.

Let a set $A$ be equivalent to the unit interval $[0,1]$. Then $A$ is said to have cardinality $c$ and to have the power of the continuum.

**Examples:** $[a,b]$, $(a,b)$, $[a,b]$, $[a,b)$, and $\mathbb{R}$ have cardinality $c$.

Let $\kappa = \#(A)$, $\tau = \#(B)$, and $A$ be equivalent to a subset of $B$. Then we write $\kappa \leq \tau$.

**Theorem 6:** The relation in the cardinal numbers defined by $\kappa \leq \tau$ is reflexive and transitive.

**Theorem 7:** (Cantor) For any set $A$, $\#(A) < \#(P(A))$ and, therefore, for any cardinal $\tau$, $\tau < 2^\tau$.

**Theorem 8:** (Schröder-Bernstein) For any cardinal numbers $\kappa, \tau$, $\kappa \leq \tau$ and $\tau \leq \kappa$ implies $\kappa = \tau$.

**Theorem 9:** (Law of Trichotomy) For any pair of cardinal numbers $\kappa$ and $\tau$, either $\kappa < \tau$, $\kappa = \tau$, or $\kappa > \tau$.

**Theorem 10:** $2^{\mathbb{N}_0} = c$.

**Continuum Hypothesis:** There does not exist cardinal number $\tau$ such that $\mathbb{N}_0 < \tau < c$. 

\[c = \mathbb{N}_0^2\]
**Generalized Continuum Hypothesis:** Let $\kappa$ be an infinite cardinal number. There does not exist cardinal number $\tau$ such that $\kappa < \tau < 2^\kappa$.

Let $\kappa$ and $\tau$ be cardinal numbers and let $A$ and $B$ be disjoint sets such that $\kappa = \#(A)$ and $\tau = \#(B)$. Then $\kappa + \tau = \#(A \cup B)$, $\kappa \tau = \#(A \times B)$, $\tau^\kappa = \#(B^A)$, where $B^A$ denotes the family of all functions from $A$ into $B$.

**Theorem 11:** $\#(2^\kappa) = \#(P(A))$.

**Theorem 12:** $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$; $(\alpha \beta) \gamma = \alpha (\beta \gamma)$; $\alpha + \beta = \beta + \alpha$; $\alpha \beta = \beta \alpha$; $\alpha(\beta + \gamma) = \alpha \beta + \alpha \gamma$; $(\alpha \beta)^\gamma = \alpha^{\beta \gamma}$; $\alpha^\beta\alpha^\gamma = \alpha^{\beta + \gamma}$; $(\alpha \beta)^\gamma = \alpha^\gamma \beta^\gamma$.

**Examples:** For natural numbers $a + b = a + c$ implies $b = c$ and $ab = ac$ implies $b = c$. But $\mathbb{N}_0 + \mathbb{N}_0 = \mathbb{N}_0 + 1$ and $\mathbb{N}_0 \neq 1$. Also $\mathbb{N}_0 \cdot \mathbb{N}_0 = \mathbb{N}_0 = \mathbb{N}_0 \cdot 1$ and $\mathbb{N}_0 \neq 1$.

**Theorem 13:** $c = \mathbb{N}^{\mathbb{N}_0}_0 = c^{\mathbb{N}_0}$; $c = c \cdot \mathbb{N}_0 = c^2 = c^{\mathbb{N}_0}$; $2^c = c^c$.

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### 6. ORDINAL NUMBERS

**Theorem 14:** Every subset of a well-ordered set is well-ordered.

**Theorem 15:** If $A$ is well-ordered and $B$ is similar to $A$, then $B$ is well-ordered.

**Theorem 16:** Let $\{A_i\}_{i \in I}$ be a well-ordered family of pairwise disjoint well-ordered sets. Then the union of the sets $\bigcup_{i \in I} A_i$ is well-ordered with the following order: Let $a, b \in \bigcup_{i \in I} A_i$; hence there exist $j, k \in I$ such that $a \in A_j$, $b \in A_k$. Now if $j < k$, $a \leq b$ and if $j = k$, then $a$ and $b$ are ordered by the ordering of $A_j$.

**Theorem 17:** All finite linearly ordered sets with the same number of elements are well-ordered and are similar to each other.

**Examples:** All permutations of a given finite set with the natural order are well-ordered and similar. The set $N$ of the natural numbers and the set $N_1 \cup N_2$ are not similar, where $N_1 = \{1, 3, 5, \ldots\}$ and $N_2 = \{2, 4, 6, \ldots\}$.

An element $b$ in a set $A$ is called an **immediate successor** of an element $a \in A$, and $a$ is called the **immediate predecessor** of $b$ if $a < b$ and there does not exist an element $c \in A$ such that $a < c < b$.

**Example:** No element in $Q$ has an immediate successor or an immediate predecessor.

**Theorem 18:** Every element in a well-ordered set (except the maximal element) has an immediate successor.

**Example:** In $N_1 \cup N_2$ 2 does not have an immediate predecessor.

An element in a well-ordered set is called a **limit element** if it does not have an immediate predecessor and if it is not the first element.

**Theorem 19:** Let $S(A) = \{S_\alpha \mid \alpha \in A\}$ ordered by $S_\alpha < S_\beta$ iff $\alpha < \beta$. Then $S(A)$ is similar to $A$ and, in particular, the function $f : A \rightarrow S(A)$ defined by $f : \alpha \rightarrow S_\alpha$ is a similarity mapping of $A$ into $S(A)$.

**Example:** $f : N \rightarrow N_2$ where $n \rightarrow 2n$ is a similarity mapping of $N$ into its subset $N_2$. Notice that for every $n \in N$, $n \leq f(n)$.

**Theorem 20:** Let $A$ be a well-ordered set, $B$ be a subset of $A$, and $f : A \rightarrow B$ be a similarity mapping of $A$ into $B$. Then for every $a \in A$, $a \leq f(a)$. 

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Corollary: Let $A$ and $B$ be similar well-ordered sets. Then there exists only one similarity mapping of $A$ into $B$.

Corollary: A well-ordered set cannot be similar to one of its initial segments.

Theorem 21: Given any two well-ordered sets, either they are similar to each other or one of them is similar to an initial segment of the other.

If a well-ordered set $A$ is equivalent to an initial segment of a well-ordered set $B$, then $A$ is said to be shorter than $B$ or $B$ is said to be larger than $A$.

Example: $N$ is shorter than $N_1 \cup N_2$.

Let $A$ be any well-ordered set and let $\lambda$ denote the family of well-ordered sets, which are similar to $A$. Then $\lambda$ is called an ordinal number and it is denoted by $\lambda = \text{ord}(A)$.

The ordinal number of each of the well-ordered sets $\emptyset$, $\{1\}$, $\{1,2\}$, $\{1,2,3\}$,... is denoted by 0, 1, 2, 3,... and is called a finite ordinal number. All other ordinals are called transfinite numbers. By definition $\omega = \text{ord}(N)$.

Let $\lambda = \text{ord}(A)$ and $\mu = \text{ord}(B)$. Then $\lambda < \mu$ if $A$ is equivalent to an initial segment of $B$, $\lambda = \mu$ if $A$ is similar to $B$, $\lambda + \mu = \text{ord}(\{A \cup B\})$, and $\lambda \cdot \mu = \text{ord}(\{A \times B\})$ where $\{A \times B\}$ is ordered reverse lexicographically i.e. $(a_1,a_2) < (a_3,b_2)$ if $b_1 < b_2$ or $b_1 = b_2$ but $a_1 < a_2$. Let $\{\lambda_i\}_{i \in I}$ be a well-ordered set of ordinal numbers such that $\lambda_i = \text{ord}(A_i)$ for some set $A_i$. Then

$$\sum_{i \in I} \lambda_i = \text{ord}\left(\bigcup_{i \in I} \{A_i \times \{i\}\}\right).$$

Theorem 22: Any set of ordinal numbers is well-ordered by $\lambda \leq \mu$.

Theorem 23: Let $S_\alpha$ be the set of ordinal numbers less than $\alpha$. Then $\alpha = \text{ord}(S_\alpha)$.

Every ordinal has an immediate successor. $\omega$ does not have immediate predecessor. Such ordinals are called limit numbers.

Theorem 24: For any ordinal $\lambda$, $\lambda + 1$ is the immediate successor of $\lambda$.

Theorem 25: $(\lambda + \mu) + \nu = \lambda + (\mu + \nu)$; $0 + \lambda = \lambda + 0$; $\lambda \cdot (\mu + \nu) = (\lambda \cdot \mu) + (\lambda \cdot \nu)$; $1 \cdot \lambda = \lambda 1 = \lambda$.

Theorem 26: $n + \omega = \omega$ but $\omega + n > \omega$, therefore the addition operation for ordinals is not commutative. $2\omega = \omega$ but $\omega 2 > \omega$, therefore the multiplication operation for ordinals is not commutative.

Theorem 27: Let $\lambda_i$, $i = 1,2,...$ is a finite ordinal greater than 0. Then $\sum_{i=1}^{\infty} \lambda_i = \omega$.

Theorem 28: (Equivalent definition of ordinal numbers) $0 = \emptyset$, $1 = \{0\}$, $2 = \{0,1\}$, $3 = \{0,1,2\}$,..., $\omega = \{0,1,2,\ldots\}$, $\omega + 1 = \{0,1,2,\ldots,\omega\}$, $\omega + 2 = \{0,1,2,...,\omega,\omega + 1\}$,..., $\omega 2 = \{0,1,2,...,\omega,\omega + 1,...\}$, $\omega 2 + 1 = \{0,1,2,...,\omega,\omega + 1,\omega 2\}$,...

Theorem 29: (Structure of ordinal numbers)

$0,1,\omega,\omega + 1,\omega + 2,...,\omega 2,\omega 2 + 1,...,\omega 3,\omega 3 + 1,...,\omega^2,\omega^2 + 1,...,\omega^2 2,...,\omega^3,...,\omega^a,...,\omega^a,...,\omega^a,...,\omega^a,...$